Lecture 5b

Matrix Inverses: algorithm

Last time

- Some but not all matrices have an inverse.
- When an inverse exists, it is unique.
- When the inverse exists, it allows us to rearrange equations.
- In particular, we can solve $Ax = b$ for x.

Goal:

- An algorithm to check invertibility and to compute inverses
- Connection between invertibility and rank

Finding inverses

A formula for the inverse of a 2×2-matrix

Consider a 2×2-matrix A, as below.

$$\mathrm{A} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- If $ad - bc \neq 0$, then A is invertible and

$$\mathrm{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

- If $ad - bc = 0$, then A is non-invertible.

The quantity $ad - bc$ is called the **determinant** of A.

We will generalize determinants to all square matrices.

Exercise 6

(i) Determine whether $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible.

(ii) If A is invertible, compute the inverse using the formula given above.

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(i) Since $ad - bc = 1 \cdot 4 - 2 \cdot 3 \neq 0$, the inverse matrix A^{-1} exists.

(ii) The formula for A^{-1} given above is

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \boxed{-\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}} = \begin{bmatrix} ? & ? \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Check: $AA^{-1} \overset{\checkmark}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \overset{\checkmark}{=} A^{-1}A$

Algorithm for finding the inverse of an $n \times n$-matrix

In general, how do we compute inverses, that is, find a solution to $\mathrm{AX} = \mathrm{Id}$?

Algorithm: Computing inverses with elementary row operations

- Write the augmented matrix $[\mathrm{A} \mid \mathrm{Id}\,]$.
- Perform elementary row operations to transform $[\mathrm{A} \mid \mathrm{Id}\,]$ into a matrix in row-echelon form (REF).
- If this REF matrix has a leading 1 in every column left of the vertical line, continue using row operations to put it in the form $[\,\mathrm{Id} \mid \mathrm{A}^{-1}]$.
- Otherwise, A is not invertible.

This method takes a lot of time and we'll learn other ways to find A^{-1}.

Exercise 7

Find, if possible, the inverse of $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, or show that it doesn't exist.

We apply the algorithm described above.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3]{2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3]{}$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3]{2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3]{}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right] \xrightarrow[-R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

Since column 3 has no leading 1, the algorithm tells us that A has no inverse.

Exercise 8

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of A, if it exists. Otherwise, show that it doesn't exist.

Using the algorithm

$$\left[\; A \;\middle|\; Id \;\right] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1]{R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

R_1

$-3R_1 + R_2$

$-R_1 + R_3$

Exercise 8

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of A, if it exists.
Otherwise, show that it doesn't exist.

Using the algorithm

$$[\ A \mid Id\ ] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\substack{R_2 \\ R_1}} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$$\begin{array}{r} R_1 \\ -3R_1+R_2 \\ -R_1+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -7 & 1 & -3 & 0 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow$$

$-3R_2+R_3$

Exercise 8

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of A, if it exists. Otherwise, show that it doesn't exist.

Using the algorithm

$$\left[\ A \mid Id\ \right] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1]{R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to$$

$$\begin{array}{r} R_1 \\ -3R_1+R_2 \\ -R_1+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -7 & 1 & -3 & 0 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \to$$

$$\begin{array}{r} \\ \\ -3R_2+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 25 & -3 & 5 & 4 \end{array}\right] \xrightarrow[\frac{1}{25}R_3]{} \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] \xrightarrow{\substack{5R_3+R_1 \\ 8R_3+R_2}}$$

The algorithm tells us that A is invertible because the columns left of the vertical line has leading 1's.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{10}{25} & 0 & -\frac{5}{25} \\ 0 & 1 & 0 & \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] = \left[\begin{array}{c|c} Id & A^{-1} \end{array}\right]$$

Therefore, A^{-1} exists, and

$$\boxed{A^{-1} = \begin{bmatrix} \frac{10}{25} & 0 & -\frac{5}{25} \\ \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 10 & 0 & -5 \\ 1 & -10 & 7 \\ -3 & 5 & 4 \end{bmatrix}}$$

Sanity check: compute AA^{-1} and $A^{-1}A$ to double check your answer.

There is a powerful connection between invertibility and rank.

Theorem (Invertibility and rank)

An $n \times n$ matrix is invertible if and only if its rank is n.

Why? (Idea)

- Start with [A | Id].
- Perform row operations to get to an REF [B | C].
- Since [B | C] is in REF, the matrix B is an REF for A.
- By def, the rank of A is the number of leading 1s of B
- So the rank of A is the number of leading 1s of [B | C] left of the vertical line.

Exercise 9

Determine whether each of the following matrices is invertible (without performing the algorithm).

$$a.) \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad b.) \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 9

Determine whether each of the following matrices is invertible (without performing the algorithm).

$$a.)\begin{bmatrix}1 & 3 & 1 & 0\\ 0 & 1 & -1 & 1\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad b.)\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

a.) The matrix is 4×4 and the rank is 4 (since the matrix is in row-echelon form and there are four leading 1s). So the matrix is invertible according to the invertibility-and-rank theorem.

b.) The matrix is 4×4 and the rank is 3 (since the matrix is in row-echelon form and there are three leading 1s). So the matrix is <u>not</u> invertible according to the theorem.

Recap Lecture 5b

- We can detect and find 2×2 inverses with an easy formula.
- We can detect and find larger inverses with a lot of work.
- We can detect invertibility from rank.

- Do required reading hw 5b + suggested pratice
- Next lecture: determinants