Lecture 5a

Matrix Inverses

Last time

How to multiply matrices by matrices.

Dangers

- AB may not equal BA.
- $AB = 0$ doesn't always imply $A = 0$ or $B = 0$.
- $AB = AC$ doesn't always imply $B = C$, even when $A \neq 0$.

As we will see, these dangers mean division doesn't always exist.

Goal

Dividing by a matrix (when it is possible).

In fact, the more elementary problem is to find inverses.

Intuition from real numbers

For real numbers, we can turn division into multiplication as long as we can find the inverse to the denominator.

$$\frac{p}{q} = \frac{1}{q}p = q^{-1}p$$

The inverse to q is the number q^{-1} such that

$$q^{-1}q = 1 \text{ and/or } qq^{-1} = 1$$

Notice that if one property is true, the other automatically is.

Let's generalize these ideas to matrices!

First, we need to generalize the number 1 to matrices.

Recall: The identity matrix

The $n \times n$ **identity matrix** Id is the $n \times n$-matrix with 1s on the diagonal and all other entries 0.

Examples

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 \end{bmatrix}$$

There is no such thing as a non-square identity matrix!

Properties of the identity matrix

For any matrix A,

$$\mathrm{Id}\ A = A \qquad A\ \mathrm{Id} = A$$

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

This is why Id is the matrix analog of the number 1.

Inverse matrices

If A is an $n \times n$-matrix, the **inverse of A** is the $n \times n$-matrix B where

$$\mathrm{AB} = \mathrm{Id} \text{ and } \mathrm{BA} = \mathrm{Id}$$

The inverse of a matrix A is usually denoted A^{-1}.

Exercise 1

Check that

$$\begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}$$

is the inverse to

$$\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix}$$

using **both** equations in the definition.

Exercise 1 (solution)

Check that $\begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}$ is the inverse to $\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix}$ using **both** equations in the definition.

$$\begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix} = \begin{bmatrix} 18\cdot 2 + -7\cdot 5 & 18\cdot -7 + -7\cdot -18 \\ 5\cdot 2 + -2\cdot 5 & 5\cdot -7 + -2\cdot -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix}\begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix} =$$

Exercise 1 (solution)

Check that $\begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}$ is the inverse to $\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix}$ using **both** equations in the definition.

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$$\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix} \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 18 + -7 \cdot 5 & 2 \cdot -7 + -7 \cdot -2 \\ 5 \cdot 18 + -18 \cdot 5 & 5 \cdot -7 + -18 \cdot -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

— end of solution —

Exercise 2: Not every matrix has an inverse!

Show that $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has no inverse.

If we could find an inverse matrix $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^{-1}$, then

$$\text{Id}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{because } A^{-1}A = \text{Id}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{because } \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ (lec } \cancel{3b}\text{ 4b)}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{multiplying by a zero matrix gives a zero matrix.}$$

So $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Impossible!

Therefore, $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ cannot have an inverse.

(This is called **proof by contradiction**, where we assume the opposite of our claim and show something impossible happens.)

"The" inverse

The inverse of a matrix is unique...if it exists!

This is why we can use unambiguous notation like A^{-1}.

If A^{-1} exists, we say the matrix A is **invertible**.
If A^{-1} doesn't exist, we say A is **non-invertible** or **not invertible**.

Left inverses and right inverses are the same

If $AB = \mathrm{Id}$ is true, then $BA = \mathrm{Id}$ is automatically true!

↳ when A and B are both square matrices!

A non-square matrix cannot have an inverse!

$\underset{2\times 3}{A}\ \underset{3\times 2}{B} = \underset{2\times 2}{\mathrm{Id}_{2\times 2}}$

$BA = \underset{3\times 3}{[\ \ ]}$

We can use inverses to rearrange equations!

Example

Assume that $AB = C$ and A is invertible. If we multiply both sides by A^{-1} on the **left**, we get

$$\begin{aligned} A^{-1}(AB) &= A^{-1}C \\ \mathrm{Id}\ B &= A^{-1}C \\ B &= A^{-1}C \end{aligned}$$

Order matters!

We must do the same thing to each side of an equation!
If $AB = C$, it would be **wrong** to assume that $A^{-1}(AB) = CA^{-1}$.

Two different kinds of division

In general, $A^{-1}C \neq CA^{-1}$. Both could be called 'C divided by A', so we avoid the terminology entirely, and we never write $\frac{C}{A}$.

Exercise 3

If B is invertible, rewrite each of the equations as formulas for A.

1. $BAB^{-1} = C$
2. $BA = -BA + CB$

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1

$$\begin{aligned} BAB^{-1} &= C \\ BAB^{-1}B &= CB \\ BA &= CB \\ B^{-1}BA &= B^{-1}CB \\ \boxed{A = B^{-1}CB} \end{aligned}$$

Sanity check: Plug in $A = B^{-1}CB$ into the original equation.

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Sanity check: Plug in $A = B^{-1}CB$ into the original equation.

2

$$\begin{aligned} BA &= -BA + CB \\ BA + BA &= CB \\ (B+B)A &= CB \\ 2BA &= CB \\ BA &= \frac{1}{2}CB \\ B^{-1}BA &= B^{-1}\left(\frac{1}{2}CB\right) \\ A &= B^{-1}\left(\frac{1}{2}CB\right) \\ A &= \boxed{\frac{1}{2}B^{-1}CB} \end{aligned}$$

Check: Plug in $A = \frac{1}{2}B^{-1}CB$ into the original equation.

Exercise 4

Let A be invertible. Check whether the inverse to $A^\top$ is $(A^{-1})^\top$.

We will verify that $A^T(A^{-1})^T = Id$ and $(A^{-1})^T A^T = Id$.

$$A^T(A^{-1})^T = (A^{-1}A)^T = Id^T = Id \checkmark$$

Lecture 4: $M^T N^T = (NM)^T$

Think: $M = A, N = A^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$$

and

Exercise 4

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We will verify that $A^T(A^{-1})^T = Id$ and $(A^{-1})^T A^T = Id$.

$$A^T(A^{-1})^T = (A^{-1}A)^T = Id^T = Id$$

and

$$(A^{-1})^T A^T = (AA^{-1})^T = Id^T = Id$$

We have shown that $(A^T)^{-1} = (A^{-1})^T$.

Properties of inverses

Assume A, B, and $A_1, A_2, \ldots, A_n$ are invertible matrices.

- $\mathrm{Id}^{-1} = \mathrm{Id}$.
- $(A^{-1})^{-1} = A$.
- $(A^n)^{-1} = (A^{-1})^n$, for example, $(AAA)^{-1} = A^{-1}\, A^{-1}\, A^{-1}$. $n=3$
- $(A^\top)^{-1} = (A^{-1})^\top$.
- $(AB)^{-1} = B^{-1}A^{-1}$.
- $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}\cdots A_2^{-1}A_1^{-1}$, *e.g,* $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ If C is invertible

Defining negative powers

If A is invertible, then for any integer n, define

$$A^n := \begin{cases} A^n & \text{if } n > 0 \\ \mathrm{Id} & \text{if } n = 0 \\ (A^{-1})^{|n|} & \text{if } n < 0 \end{cases}$$

$A^{-4} = A^{-1}.A^{-1}.A^{-1}.A^{-1}$

$A^{-2}A = A^{-2+1} = A^{-1}$

Then, for any integers m and n, $A^mA^n = A^{m+n}$.

Solving systems of linear equations with inverses

Suppose that a system of n linear equations in n variables is written in matrix form as $A\vec{x} = \vec{b}$. If A is invertible, then this system has a unique solution, given by

$$\vec{x} = A^{-1}\vec{b}$$

If A is non-invertible, then we can't say anything yet.

Exercise 5

Solve the system of linear equations

$$\begin{aligned} 2x - 7y &= 3 \\ 5x - 18y &= 8 \end{aligned}$$

using inverses.

Systems of Linear Equations and Inverses

Exercise 5 (solution)

(Step i) Turn the following system of linear equations into a matrix equation of the form $A\vec{x} = \vec{b}$.

$$\begin{aligned} 2x - 7y &= 3 \\ 5x - 18y &= 8 \end{aligned}$$

The matrix equation in the form $A\vec{x} = \vec{b}$ is

$$\underbrace{\overset{A}{\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix}}}_{\text{coefficients}} \underbrace{\overset{\vec{x}}{\begin{bmatrix} x \\ y \end{bmatrix}}}_{\text{variables}} = \underbrace{\overset{\vec{b}}{\begin{bmatrix} 3 \\ 8 \end{bmatrix}}}_{\text{constant terms}}$$

Exercise 5 (solution)

(Step ii) Solve by computing $\mathrm{A}^{-1}\vec{b}$ (we already computed A^{-1} in Exercise 1).

Since A^{-1} exists and has the property $\mathrm{A}^{-1}\mathrm{A} = \mathit{Id}$ we obtain the following.

$$
\begin{aligned}
\mathrm{A}\vec{x} &= \vec{b} \\
\mathrm{A}^{-1}(\mathrm{A}\vec{x}) &= \mathrm{A}^{-1}\vec{b} \\
(\mathrm{A}^{-1}\mathrm{A})\vec{x} &= \mathrm{A}^{-1}\vec{b} \\
\mathrm{Id}\ \vec{x} &= \mathrm{A}^{-1}\vec{b} \\
\vec{x} &= \mathrm{A}^{-1}\vec{b}
\end{aligned}
$$

Exercise 5 (solution)

(Step ii) Solve by computing $\mathsf{A}^{-1}\vec{b}$ (we already computed A^{-1} in Exercise 1).

Since A^{-1} exists and has the property $\mathsf{A}^{-1}\mathsf{A} = I$, we obtain the following.

$$\begin{aligned} \mathsf{A}\vec{x} &= \vec{b} \\ \mathsf{A}^{-1}(\mathsf{A}\vec{x}) &= \mathsf{A}^{-1}\vec{b} \\ (\mathsf{A}^{-1}\mathsf{A})\vec{x} &= \mathsf{A}^{-1}\vec{b} \\ \mathrm{Id}\ \vec{x} &= \mathsf{A}^{-1}\vec{b} \\ \vec{x} &= \mathsf{A}^{-1}\vec{b} \end{aligned}$$

i.e., $\mathsf{A}\vec{x} = \vec{b}$ has the unique solution given by $\vec{x} = \mathsf{A}^{-1}\vec{b}$. Therefore,

$$\vec{x} = A^{-1}\begin{bmatrix} 3 \\ 8 \end{bmatrix} \underset{\text{by Ex. 1}}{=} \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ -1 \end{bmatrix}}$$

Exercise 5 (solution)

(Step iii) After computing $A^{-1}\vec{b}$, and plug it back in the system.

Sanity check: verify that $x = -2$, $y = -1$ is a solution to the system (plug in).

$$\begin{aligned} 2(-2) - 7(-1) &= 3 \\ 5(-2) - 18(-1) &= 8 \end{aligned}$$

Recap Lecture 5a

- Some but not all square matrices have an inverse.
- When an inverse exists, it is unique.
- When the inverse exists, it allows us to rearrange equations.
- In particular, we can solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

Next time: How to determine when the inverse exists and how to compute it.

Do suggested practice