

Lecture 4b

Matrix Multiplication (properties of matrix arithmetic)

Do suggested
practice
See "link to
lecture notes"

Upcoming:

- Reading HW 4b (Sun)
- Worksheet (Sun)

Properties of matrix arithmetic

Last Time

Matrix multiplication violates one of the basic rules of traditional arithmetic: **matrix multiplication does not always commute.**

Goal

What familiar properties **does** matrix arithmetic have?

Matrix addition is associative and commutative

Associativity of addition

$$A + (B + C) = (A + B) + C$$

So, we can write $A + B + C$ to mean

“ $A + (B + C)$ or $(A + B) + C$, your choice”

Commutativity of addition

$$A + B = B + A$$

As a result, we can rearrange sums of matrices however we want.

$$\begin{aligned} A + B + C &= C + A + B \\ &= B + C + A \\ &= C + B + A \end{aligned}$$

Multiplication distributes over addition

$$A(B + C) = AB + AC \quad (B + C)A = BA + CA$$

Exercise 4

Compute $Av + Aw$, where $A := \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$, $v := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $w := \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Note: It takes less effort to compute $A(v + w)$ than to compute $Av + Aw$.

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$$(B + C)A = BA + CA$$

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Note: It takes less effort to compute $A(v + w)$ than to compute $Av + Aw$.

$$v + w = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$A(v + w) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + 1 \cdot 3 \\ -1 \cdot 3 + 1 \cdot 3 \\ 3 \cdot 3 + 0 \cdot 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix}$$

Handwritten notes: Dimensions are circled in pink: 3×2 and 2×1 . The result vector is circled in green. The calculation steps are boxed in blue, red, and green.

Matrix multiplication is associative

Associativity of multiplication

$$A(BC) = (AB)C$$

So, we can write ABC to mean “ $A(BC)$ or $(AB)C$, your choice”.

Exercise 5

Check that $(AB)C = A(BC)$, where

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$B := \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & -1 & 2 \end{bmatrix}$$

$$C := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Compute $A(BC)$ and $(AB)C$. Which took more computation?

So far

- Matrix addition is both associative and commutative
- Matrix multiplication is associative but is not commutative

Can you think of a familiar arithmetic operation (from elementary school) which is not associative?

Transpose reverses the order of multiplication

$$(AB)^{\top} = B^{\top}A^{\top}$$

Exercise 6

Check that $(AB)^{\top} = B^{\top}A^{\top}$ for $A := \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$, $B := \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$

Exercise 6

Check that $(AB)^T = B^T A^T$ for $A := \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}$, $B := \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 1 \cdot 2 & 2 \cdot (-1) + 1 \cdot 2 \\ -1 \cdot 1 + 1 \cdot 2 & -1 \cdot (-1) + 1 \cdot 2 \\ 3 \cdot 1 + 0 \cdot 2 & 3 \cdot (-1) + 0 \cdot 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 3 \\ 3 & -3 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

3rd row is now 3rd col

$$B^T = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \begin{matrix} (1,1) \\ 1 \cdot 2 + 2 \cdot 1 \end{matrix} & \begin{matrix} (1,2) \\ 1 \cdot (-1) + 2 \cdot 1 \end{matrix} & \begin{matrix} (1,3) \\ 1 \cdot 3 + 2 \cdot 0 \end{matrix} \\ \begin{matrix} (2,1) \\ -1 \cdot 2 + 2 \cdot 1 \end{matrix} & \begin{matrix} (2,2) \\ -1 \cdot (-1) + 2 \cdot 1 \end{matrix} & \begin{matrix} (2,3) \\ -1 \cdot 3 + 2 \cdot 0 \end{matrix} \end{bmatrix} = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 3 & -3 \end{bmatrix}$$

2×2 2×3 2×3

Theorem

Suppose A and B are matrices.

- 1 $(A^T)^T = A$.
- 2 $(rA)^T = r(A^T)$ if r is a number.
- 3 $(A + B)^T = A^T + B^T$.

Algebra with matrices

Using these rules, we can rearrange expressions involving matrices.

Exercise 7

Solve the following expression for B . Write B in terms of A .

$$(A^T + 2B^T)^T = 2A$$

Theorem

- 1 $(A^T)^T = A.$
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Exercise 7 (solution)

$(A^T + 2B^T)^T = 2A$, the original equation

$$A^T + 2B^T = (2A)^T \text{ by part (1)}$$

$$A^T + 2B^T = 2(A^T) \text{ by part (2) or (3)}$$

$$2B^T = A^T, \text{ after adding } -A^T \text{ to both sides}$$

$$B^T = \frac{1}{2}(A^T) = \left(\frac{1}{2}A\right)^T \text{ by part (2)}$$

$$B = \boxed{\frac{1}{2}A} \text{ by part (1)}$$

To verify your solution, plug in $B = \frac{1}{2}A$ back into the original equation.

Let's return to the properties that **aren't** true.

Recall: Matrix multiplication is not commutative

AB may not equal BA.

So, although we can pick the order in which we multiply pairs of neighbors (due to associativity), for example,

$$ABCD = A((BC)D)$$

we cannot rearrange the matrices.

ABCD may not equal ADCB

Example

Convince yourself that

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Warning: The product of two non-zero matrices might be zero

$AB = 0$ doesn't always imply $A = 0$ or $B = 0$.

This never happens for numbers: if $ab = 0$, then $a = 0$ or $b = 0$.

There's a more general version of this phenomenon.

Warning: Multiplication can't always be cancelled

Having $AB = AC$ and $A \neq 0$ doesn't guarantee $B = C$.

This never happens for numbers: if $ab = ac$ and $a \neq 0$, then $b = c$.

Exercise 8

Compute the following.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

There's a more general version of this phenomenon.

Warning: Multiplication can't always be cancelled

Having $AB = AC$ and $A \neq 0$ doesn't guarantee $B = C$.

This never happens for numbers: if $ab = ac$ and $a \neq 0$, then $b = c$.

Exercise 8

Compute the following.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + -1 \cdot 2 \\ -1 \cdot 1 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot -2 + -1 \cdot -1 \\ -1 \cdot -2 + 1 \cdot -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Recap

- Some familiar properties of arithmetic hold for matrices.
- Some familiar properties don't!

...wait, what happened to division?

Next time

Why division doesn't always work, and how to perform division when it does work.