Lecture 4a

Matrix Multiplication

For practice problems + key, see the "Lecture notes"

Upcoming:

- Reading HW 4a (Thurs)
- In-class Quiz (Fri)
- Worksheet (Sun)

Recall from lecture 3b: Multiplying a matrix and a vector

Given a $m \times n$-matrix A and an n-vector v, Av is the m-vector whose ith entry is the dot product of the ith row of A with v.

Example: Multiplying a 3×3 matrix and a 3-vector

$$\underbrace{\begin{bmatrix} 2 & -3 & 4 \\ -2 & 2 & -3 \\ -3 & 4 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}_{v} = \begin{bmatrix} 2 \cdot 2 + (-3) \cdot 1 + 4 \cdot (-1) \\ -2 \cdot 2 + 2 \cdot 1 - 3 \cdot (-1) \\ -3 \cdot 2 + 4 \cdot 1 + 2 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$$

Goal for lecture 4a

Multiplying a matrix by a marix.

Denoting a specific entry of a matrix

The **(i, j)th entry of a matrix** A is the entry in the ith row and the jth column, and is denoted $A_{i,j}$ or $a_{i,j}$.

Example

The $(2, 3)$th entry of the following matrix is **2**.

$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & \mathbf{2} & 0 \end{bmatrix}$$

This notation counts down then over, unlike Cartesian coordinates.

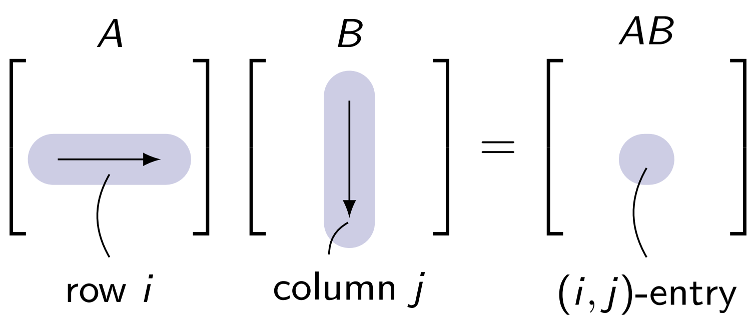
$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4 \\ (2,1) & (2,2) & \mathbf{(2,3)} & (2,4) \end{bmatrix}$$

Matrix multiplication: the idea

Given matrices A and B, the product AB is the matrix whose (i, j)th entry is the dot product of row i of A and column j of B.

To compute the (i, j)-entry of AB, do:

Go *across* row i of A, and *down* column j of B, multiply corresponding entries, and add the results.



The rows of A must be the same length as the columns of B.

Matrix multiplication: the formula

The product AB is a matrix whose (i,j)th entry is the sum of the product of the row i of A with the column j of B.

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cdots & b_{1,j} & \cdots \\ \cdots & b_{2,j} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & b_{n,j} & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \ddots & \vdots & \ddots \\ \cdots & a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,n}b_{n,j} & \cdots \\ \ddots & \vdots & \ddots \end{bmatrix}$$

Exercise 1

Compute the $(1,3)$- and $(2,4)$-entries of AB where

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}.$$

Then compute AB.

The $(1,3)$th-entry of AB is the dot product of row 1 of A and column 3 of B (highlighted in the following display), computed by multiplying corresponding entries and adding the results.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}$$

$$(1,3)\text{-entry} = 3 \cdot 6 + (-1) \cdot 3 + 2 \cdot 5 = 25$$

$$\begin{bmatrix} \cdot & \cdot & \mathbf{25} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Similarly, the $(2,4)$-entry of AB involves row 2 of A and column 4 of B.

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}$$

$$(2,4)\text{-entry} = 0 \cdot 0 + 1 \cdot 4 + 4 \cdot 8 = 36$$

$$\begin{bmatrix} \cdot & \cdot & 25 & \cdot \\ \cdot & \cdot & \cdot & \mathbf{36} \end{bmatrix}$$

Pause the video and compute the rest of the entries of AB.

$$\begin{array}{cccccc} & A & & B & & AB \\ AB = & \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} & & \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} & = & \begin{bmatrix} 4 & 1 & 25 & 12 \\ -4 & 2 & 23 & 36 \end{bmatrix} \\ \text{Dimensions} & 2 \times 3 & & 3 \times 4 & & 2 \times 4 \end{array}$$

Dimensions must match!

For AB to exist, we need

$$\mathrm{width}(\mathsf{A}) = \mathrm{height}(\mathsf{B})$$

Size of a product

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

It can be helpful to think of the middle dimensions as cancelling:

$$(m \times \cancel{n)(n} \times p) = m \times p$$

If the middle dimensions don't agree, the product doesn't exist!

Exercise 2(i)

Compute the product

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix}$$

Exercise 2(i) (solution)

Compute the product

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 1.2 + 2.-1 + -1.0 & 1.1 + 2.3 + (-1).1 \\ 1.2 + 0.-1 + 2.0 & 1.1 + 0.3 + 2.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 \\ 2 & 3 \end{bmatrix}$$

	A	B	AB
Dimensions	2×3	3×2	2×2

Exercise 2(ii)

Compute the product

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

Exercise 2(ii) (solution)

Compute the product

$$\underset{\substack{B \\ 3\times 2}}{\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 1 \end{bmatrix}} \underset{\substack{A \\ 2\times 3}}{\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}} =$$

$$= \begin{bmatrix} 3 & 4 & 0 \\ 2 & -2 & 7 \\ 1 & 0 & 2 \end{bmatrix}$$

AB

3×3

Order matters in matrix multiplication!

We have seen that

$$\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & 3\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 6\\ 2 & 3\end{bmatrix} \neq \begin{bmatrix}2 & 1\\ -1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 2\end{bmatrix}$$

This means that matrix multiplication is not always commutative!

That is, AB is usually not equal to BA, though it can happen.

Exercise 3

Check whether the following matrices commute.

$$\begin{bmatrix}1 & -2\\ 1 & 2\end{bmatrix} \text{ and } \begin{bmatrix}1 & 6\\ -3 & -2\end{bmatrix}$$

Order matters in matrix multiplication!

Matrix multiplication is not always commutative!

That is, AB is usually not equal to BA, though it can happen.

Exercise 3 (solution)

Check whether the following matrices commute.

$$\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 6 \\ -3 & -2 \end{bmatrix}$$

Answer: Yes, they commute

$$\begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + -2 \cdot (-3) & 1 \cdot 6 + -2 \cdot -2 \\ 1 \cdot 1 + 2 \cdot -3 & 1 \cdot 6 + 2 \cdot -2 \end{bmatrix} = \begin{bmatrix} \boxed{7} & \boxed{10} \\ \boxed{-5} & \boxed{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 6 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 6 \cdot 1 & 1 \cdot -2 + 6 \cdot 2 \\ -3 \cdot 1 + -2 \cdot 1 & -3 \cdot -2 + -2 \cdot 2 \end{bmatrix} = \begin{bmatrix} \boxed{7} & \boxed{10} \\ \boxed{-5} & \boxed{2} \end{bmatrix}$$