Lecture 3b

Matrices and Vectors: matrix-vector multiplication

Upcoming:

Reading HW 3b

Last Time

- Matrices and vectors
- Matrix addition
- Scalar multiplication
- Linear combinations (combinations of the operations above)
- Transposes

Plan

- Zero matrices
- Matrix multiplication : multiplying a matrix & a vector
- Identity matrices

Zero matrices

Zero matrices

A **zero matrix** is a matrix whose entries are all zero.

Confusingly, people usually denote these by $\mathbf{0}$ and hope the size is clear.

Example

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $A + 0$ means $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $= \begin{bmatrix} 1+0, & 2+0, & 3+0 \\ 4+0, & 5+0, & 6+0 \end{bmatrix}$

Adding zero doesn't change anything

As long as A and 0 have the same size, $A + 0 = A$.

Multiplying a matrix and a vector: Motivation

Next up:

Multiplying a matrix and a vector.

Motivating question

How can we check a potential solution of a linear system, directly from the the **augmented matrix**?

Let's recap a familiar computation.

Recap: checking a solution

Consider the linear system

$$\begin{aligned} 2x - 3y + 4z &= -3 \\ -2x + 2y - 3z &= 1 \\ -3x + 4y + 2z &= -4 \end{aligned}$$

To check whether $x = 2$, $y = 1$, $z = -1$ is a solution, we plug in:

$$\begin{aligned} 2 \qquad - 3 \qquad + 4 \cdot (\qquad) &= \qquad \checkmark \\ -2 \cdot \qquad + 2 \qquad - 3 \cdot \qquad) &= \qquad \checkmark \\ -3 \qquad + 4 \qquad + 2 \cdot (\qquad) &= \qquad \checkmark \end{aligned}$$

Let's recap a familiar computation.

Recap: checking a solution

Consider the linear system

$$\begin{aligned} 2x - 3y + 4z &= -3 \\ -2x + 2y - 3z &= 1 \\ -3x + 4y + 2z &= -4 \end{aligned}$$

To check whether $x = 2$, $y = 1$, $z = -1$ is a solution, we plug in:

$$\begin{aligned} 2 \cdot 2 - 3 \cdot 1 + 4 \cdot (-1) &= 4 - 3 - 4 = -3 \quad \checkmark \\ -2 \cdot 2 + 2 \cdot 1 - 3 \cdot (-1) &= -4 + 2 + 3 = 1 \quad \checkmark \\ -3 \cdot 2 + 4 \cdot 1 + 2 \cdot (-1) &= -6 + 4 - 2 = -4 \quad \checkmark \end{aligned}$$

Therefore, $(x, y, z) = (2, 1, -1)$ is a solution.

To check a solution, it will be helpful to split the augmented matrix

$$\begin{aligned} 2x - 3y + 4z &= -3 \\ -2x + 2y - 3z &= 1 \\ -3x + 4y + 2z &= -4 \end{aligned} \quad \mapsto \left[\begin{array}{ccc|c} 2 & -3 & 4 & -3 \\ -2 & 2 & -3 & 1 \\ -3 & 4 & 2 & -4 \end{array}\right]$$

...into a matrix A of coefficients and a vector b of constants:

$$\text{matrix of coefficients} = \underbrace{\begin{bmatrix} 2 & -3 & 4 \\ -2 & 2 & -3 \\ -3 & 4 & 2 \end{bmatrix}}_{A}, \quad \text{vector of constants} = \underbrace{\begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}}_{\mathrm{b}}$$

The potential solution can also be collected into a vector:

Potential Solution $x=2,\ y=1,\ z=-1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{matrix of coefficients} = \underbrace{\begin{bmatrix} 2 & -3 & 4 \\ -2 & 2 & -3 \\ -3 & 4 & 2 \end{bmatrix}}_{A}, \quad \text{vector of constants} = \underbrace{\begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}}_{b}$$

Each row of A represents one equation. Let's focus on the first row.

Plugging into a single formula

Plugging $x = 2, y = 1, z = -1$ into the first formula

$$2x - 3y + 4z \left(= 2 \cdot 2 - 3 \cdot 1 + 4 \cdot (-1)\right)$$

becomes the following

$$\begin{bmatrix} 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 2 \cdot 2 + (-3) \cdot 1 + 4 \cdot (-1) = -3$$

We can plug into all three equations at the same time.

Plugging into several formulas

Plugging $x = 2, y = 1, z = -1$ into the formulas

$$2x - 3y + 4z$$

$$-2x + 2y - 3z$$

$$-3x + 4y + 2z$$

can be translated into the following computation.

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} & + & & + & \\ & + & & - & \\ & + & & + & \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

We can plug into all three equations at the same time.

Plugging into several formulas

Plugging $x = 2, y = 1, z = -1$ into the formulas

$$2x - 3y + 4z$$
$$-2x + 2y - 3z$$
$$-3x + 4y + 2z$$

can be translated into the following computation.

$$\begin{bmatrix} 2 & -3 & 4 \\ -2 & 2 & -3 \\ -3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + (-3) \cdot 1 + 4 \cdot (-1) \\ -2 \cdot 2 + 2 \cdot 1 - 3 \cdot (-1) \\ -3 \cdot 2 + 4 \cdot 1 + 2 \cdot (-1) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

Multiplying a matrix and a vector: Definition

Definition: Multiplying a matrix and a vector

A matrix A and vector v can by multiplied if

$$\mathrm{width}(\mathrm{A}) = \mathrm{height}(\mathrm{v})$$

The product Av is a vector whose ith entry is the sum of the product of the entries of the ith row of A with the entries of v.

$$\begin{array}{c} \\ \\ \text{row } i \\ \\ \end{array} \overbrace{\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}}^{\text{width}} \left.\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right\} \text{height of } v = \begin{bmatrix} \vdots \\ a_{i,1}v_1 + a_{i,2}v_2 + \ldots + a_{i,n}v_n \\ \vdots \end{bmatrix} \begin{array}{l} \\ \leftarrow i\text{-th entry} \\ \\ \end{array}$$

If $\mathrm{width}(\mathrm{A}) \neq \mathrm{height}(\mathrm{v})$, then the product Av doesn't make sense!

(undefined)

Exercise 3

Compute the following products.

a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Exercise 3 (solution)

Compute the following products.

a) $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot -1 + 3\cdot 1 \\ 4\cdot 1 + 5\cdot -1 + 6\cdot 1 \\ 7\cdot 1 + 8\cdot -1 + 9\cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + -1\cdot 2 + 2\cdot 3 + -2\cdot 4 \\ 0\cdot 1 + 1\cdot 2 + 2\cdot 3 + 0\cdot 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$$

Sizes in matrix-vector multiplication

The product of an $m \times n$-matrix and an n-vector is an m-vector.

height, # of rows

width, # of columns

must be the same as the width of the matrix

will always be the same as the height of the matrix

$$\begin{bmatrix} \circ & \circ \\ \circ & \circ \\ \circ & \circ \\ \circ & \circ \end{bmatrix} \begin{bmatrix} \square \\ \square \end{bmatrix} = \begin{bmatrix} \star \\ \star \\ \star \\ \star \end{bmatrix}$$

$\underset{m \quad n}{4 \times 2}$ $\quad \underset{n}{2 \times 1}$ $\quad \underset{m}{4 \times 1}$

Trick to remember:

$$4 \times \cancel{2} \ \cancel{2} \times 1$$
$$\Downarrow$$
$$4 \times 1$$

$$m \times \cancel{n} \ \cancel{n} \times 1$$
$$\Downarrow$$
$$m \times 1$$

Turning a linear system into a matrix equation

Consider a linear system, and let

- A be the matrix of coefficients,
- b be the vector of constants, and
- x be the vector of variables.

Then the linear system can be rewritten as the vector equation

$$\mathrm{A}\mathrm{x} = \mathrm{b}$$

Example

$$\begin{aligned} 2x - 3y + 4z &= -3 \\ -2x + 2y - 3z &= 1 \\ -3x + 4y + 2z &= -4 \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} 2 & -3 & 4 \\ -2 & 2 & -3 \\ -3 & 4 & 2 \end{bmatrix}}_{\mathrm{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathrm{x}} = \underbrace{\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}}_{\mathrm{b}}$$

Exercise 4

Check that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

is a solution to the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 4

Check that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

is a solution to the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot -2 + 3 \cdot 1 \\ 4 \cdot 1 + 5 \cdot -2 + 6 \cdot 1 \\ 7 \cdot 1 + 8 \cdot -2 + 9 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

The answer is $\boxed{\text{Yes}}$.

Identity matrices

For a positive integer n, the $n \times n$ **identity matrix** is the $n \times n$ matrix whose diagonal entries are 1 and whose other entries are 0.

People often denote identity matrices by Id, again hoping the size is clear.

Multiplying by an identity matrix doesn't change a vector (assuming the sizes are such that the product is defined).

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \boxed{1 \cdot 4} + 0 \cdot -1 + 0 \cdot 2 \\ 0 \cdot 4 + \boxed{1 \cdot -1} + 0 \cdot 2 \\ 0 \cdot 4 + 0 \cdot -1 + \boxed{1 \cdot 2} \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

Identity matrices must be **square**; that is, its height equals its width.

Recap

- Zero matrices
- Matrix-vector multiplication
- Identity matrices

Next time

How and why to multiply two matrices.