Matrices and Vectors: matrix-vector multiplication

Upcoming:

Reading HW 3b

Last Time

Zero matrices

- Matrices and vectors
- Matrix addition
- Scalar multiplication
- Linear combinations (combinations of the operations above)
- Transposes

Plan

- Zero matrices
- Matrix multiplication: multiplying a matrix & a vector
- Identity matrices

Zero matrices

Zero matrices

A zero matrix is a matrix whose entries are all zero.

Confusingly, people usually denote these by ${\bf 0}$ and hope the size is clear.

Example

If
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
, then $A + 0$ means $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 + 0, 2 + 0, 3 + 0 \\ 4 + 0, 5 + 0, 6 + 0 \end{bmatrix}$

Adding zero doesn't change anything

As long as A and 0 have the same size, A + 0 = A.

Multiplying a matrix and a vector: Motivation

Next up:

Multiplying a matrix and a vector.

Motivating question

How can we check a potential solution of a linear system, directly from the **augmented matrix**?

Let's recap a familiar computation.

Recap: checking a solution

Consider the linear system

$$2x - 3y + 4z = -3$$
$$-2x + 2y - 3z = 1$$
$$-3x + 4y + 2z = -4$$

To check whether x = 2, y = 1, z = -1 is a solution, we plug in:

$$2 - 3 + 4 \cdot () =$$
 $-2 + 2 - 3 \cdot) =$
 $-3 + 4 + 2 \cdot () =$

Let's recap a familiar computation.

Recap: checking a solution

Consider the linear system

$$2x - 3y + 4z = -3$$
$$-2x + 2y - 3z = 1$$
$$-3x + 4y + 2z = -4$$

To check whether x = 2, y = 1, z = -1 is a solution, we plug in:

$$2 \cdot 2 - 3 \cdot 1 + 4 \cdot (-1) = 4 - 3 - 4 = -3$$

 $-2 \cdot 2 + 2 \cdot 1 - 3 \cdot (-1) = -4 + 2 + 3 = 1$
 $-3 \cdot 2 + 4 \cdot 1 + 2 \cdot (-1) = -6 + 4 - 2 = -4$

Therefore, (x, y, z) = (2, 1, -1) is a solution.

To check a solution, it will be helpful to split the augmented matrix

$$2x - 3y + 4z = -3
-2x + 2y - 3z = 1 \mapsto$$

$$-3x + 4y + 2z = -4$$

$$\begin{bmatrix}
2 & -3 & 4 & -3 \\
-2 & 2 & -3 & 1 \\
-3 & 4 & 2 & -4
\end{bmatrix}$$

...into a matrix A of coefficients and a vector b of constants:

matrix of coefficients
$$= \underbrace{\begin{bmatrix} 2 & -3 & 4 \\ -2 & 2 & -3 \\ -3 & 4 & 2 \end{bmatrix}}_{A}, \text{ vector of constants} = \underbrace{\begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}}_{B}$$

The potential solution can also be collected into a vector:

Potential
$$x=2, y=1, z=-1$$

$$Solution$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

matrix of coefficients =
$$\begin{bmatrix} 2 & -3 & 4 \\ -2 & 2 & -3 \\ -3 & 4 & 2 \end{bmatrix}$$
, vector of constants =
$$\begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

Each row of A represents one equation. Let's focus on the first row.

Plugging into a single formula

Plugging x = 2, y = 1, z = -1 into the first formula

$$2x - 3y + 4z = 2.2 - 3.1 + 4.(-1)$$

becomes the following

$$\begin{bmatrix} 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 2 \cdot 2 + (-3) \cdot 1 + 4 \cdot (-1) = -3$$

We can plug into all three equations at the same time.

Plugging into several formulas

Plugging x = 2, y = 1, z = -1 into the formulas

$$2x - 3y + 4z$$

$$-2x + 2y - 3z$$

$$-3x + 4y + 2z$$

can be translated into the following computation.

We can plug into all three equations at the same time.

Plugging into several formulas

Plugging
$$x = 2$$
, $y = 1$, $z = -1$ into the formulas $2x - 3y + 4z$

$$-2x + 2y - 3z$$

$$-3x + 4y + 2z$$

can be translated into the following computation.

$$\begin{bmatrix} 2 & -3 & 4 \\ -2 & 2 & -3 \\ -3 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 + (-3) \cdot 1 + 4 \cdot (-1) \\ -2 \cdot 2 + 2 \cdot 1 - 3 \cdot (-1) \\ -3 \cdot 2 + 4 \cdot 1 + 2 \cdot (-1) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

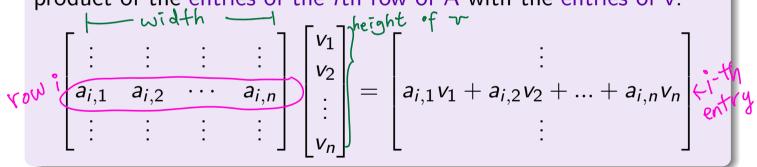
Multiplying a matrix and a vector: Definition

Definition: Multiplying a matrix and a vector

A matrix A and vector v can by multiplied if

$$\operatorname{width}(A) = \operatorname{height}(v)$$

The product Av is a vector whose ith entry is the sum of the product of the entries of the ith row of A with the entries of v.



If width(A) \neq height(v), then the product Av doesn't make sense!

Exercise 3

Compute the following products.

a)
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

b)
$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Exercise 3 (solution)

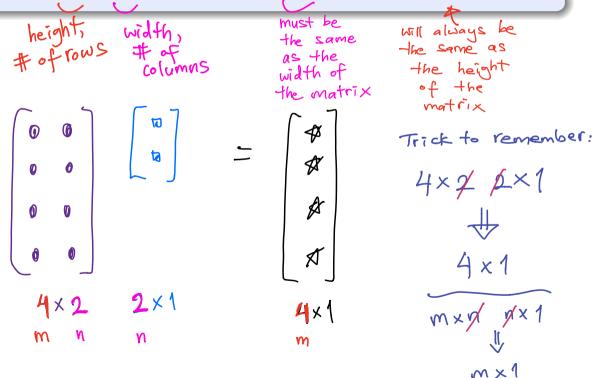
Compute the following products.

a)
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ \hline 7 & 8 & 9 \end{bmatrix}$$
 $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ =

b)
$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Sizes in matrix-vector multiplication

The product of an $m \times n$ -matrix and an n-vector is an m-vector.



Turning a linear system into a matrix equation

Consider a linear system, and let

- A be the matrix of coefficients,
- b be the vector of constants, and
- x be the vector of variables.

Then the linear system can be rewritten as the vector equation

$$Ax = b$$

Example

$$2x - 3y + 4z = -3
-2x + 2y - 3z = 1
-3x + 4y + 2z = -4$$

$$\begin{bmatrix}
2 & -3 & 4 \\
-2 & 2 & -3 \\
-3 & 4 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
3 \\
1 \\
-4
\end{bmatrix}$$

Check that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

is a solution to the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exercise 4

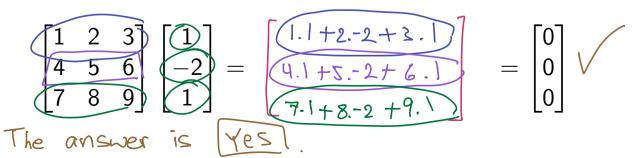
Check that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

is a solution to the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:



Identity matrices

For a positive integer n, the $n \times n$ identity matrix is the $n \times n$ matrix whose diagonal entries are 1 and whose other entries are 0.

People often denote identity matrices by Id, again hoping the size is clear.

Multiplying by an identity matrix doesn't change a vector (assuming the sizes are such that the product is defined).

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.4 + 0.1 + 0.2 \\ 0.4 + 1.1 + 0.2 \\ 0.4 + 0.1 + 1.2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

Identity matrices must be **square**; that is, its height equals its width.

Zero matrices

- Matrix-vector multiplication
- Identity matrices

Next time

How and why to multiply two matrices.

Identity matrices