

Matrices and Vectors

Upcoming (due Thurs):

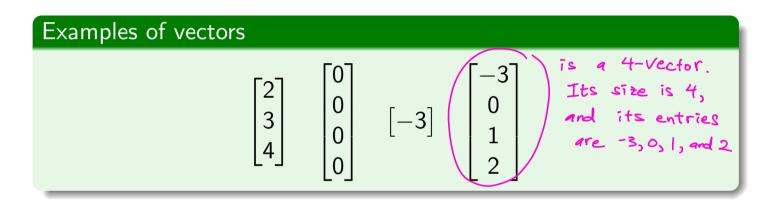
Reading HW 3a

Plan

Introduce vectors, matrices, and matrix arithmetic.

Definition: Vectors

A vector is a column of numbers, surrounded by brackets.



- The entries are sometimes called **coordinates**.
- The size of a vector is the number of entries.
- An *n*-vector is a vector of size *n*.
- We sometimes call these **column vectors**, to distinguish them from **row vectors** (when we write the numbers horizontally).

An example of a row vector: [2, 3, 4]

The purpose of vectors

The purpose of vectors is to collect related numbers together and work with them as a single object.

Example of a vector: Latitude and longitude

A point on the globe can be described by two numbers: the latitude and longitude. These can be combined into a single vector:

Position of Norman's train station = $[35.13124^\circ, -97.26343^\circ]$

(row vector

Example of a vector: solution of a linear system

A **solution** of a linear system is given in terms of values of variables, even though we think of this as one object:

x = 3, y = 0, z = -1(x, y, z) = (3, 0, -1)

We can restate this as a (column) vector:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$$

Definition: Matrices

A matrix is a rectangular grid of numbers, surrounded by brackets.

Examples of matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

• The individual numbers are called the entries of the matrix.

#rows #cols

- The **height** of a matrix is the number of **rows**.
- The width of a matrix is the number of columns.
- An $m \times n$ -matrix is a matrix of height m and width n.

The purpose of matrices

Matrices serve two distinct purposes.

- 1 To collect tables of related numbers together.
- **2** To describe linear transformations between vector spaces.

The second item should be totally mysterious right now!

Example: Several global positions

We can collect the latitude and longitude of several locations (e.g. Dale Hall, PHSC, and the Union) into a single matrix.

 $\begin{bmatrix} 35.204183^{\circ} & 35.209397^{\circ} & 35.209474^{\circ} \\ -97.446405^{\circ} & -97.447460^{\circ} & -97.444157^{\circ} \end{bmatrix}$

Example: Economic mobility

We can make a matrix whose entries record the probability that the child born in the upper/middle/lower income bracket will grow up into the upper/middle/lower income bracket. ^a

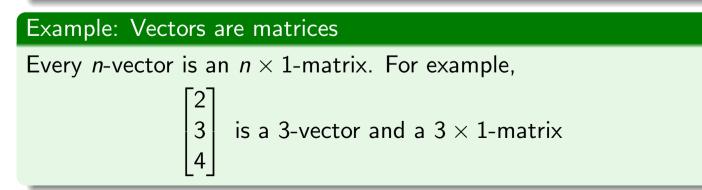
 50%
 20%
 20%

 40%
 60%
 30%

 10%
 20%
 50%

For example, being born in the upper bracket gives a 10% chance of ending up in the lower bracket.

^aThese nice whole numbers are from Exercise 2.9.4 in the textbook, not based on actual data



Arithmetic with vectors and matriceslinear combinationTranspose of a matrix•000000

Arithmetic with vectors and matrices

We define several arithmetic operations on matrices.

The sum of two matrices

If A and B are matrices of the same size, then A + B is the matrix whose entries add the corresponding entries of A and B.

Variables denoting matrices are usually capitalized.

Example

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 2 & 1 \\ 5 & 5 & 5 & 8 \end{bmatrix}$$

The sum of two matrices of different sizes does not make sense!

Arithmetic with vectors and matriceslinear combinationTranspose of a matrix000000

Arithmetic with vectors and matrices

We define several arithmetic operations on matrices.

The sum of two matrices

If A and B are matrices of the same size, then A + B is the matrix whose entries add the corresponding entries of A and B.

Example

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & -2 \end{bmatrix}$$

(2×4 matrix) (2×3 matrix)

is not defined.

Scalar multiplication

If A is a matrix and r is a number, then rA is the matrix whose entries multiply the corresponding entry of A by r.

By 'number', we mean a real number unless otherwise specified.

Example $2 \cdot \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -2 & -6 \\ 0 & -2 & -4 & 0 \end{bmatrix}$

Scalar is just a fancy word for 'number'. We are distinguishing scalar multiplication from matrix multiplication, which we'll learn about in another lecture.

Scalar multiplication distributes over addition:

r(A+B) = rA + rB

Example:

$$2 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 2 & 1 \\ 5 & 5 & 5 & 8 \end{bmatrix}$$

$$2 \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} + 2 \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -1 & -2 & 0 \end{bmatrix}$$



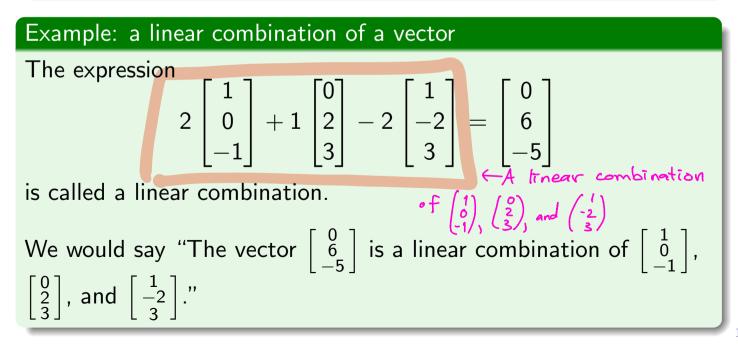
We combine addition and scalar multiplication into one concept.

Linear combinations

A linear combination of matrices is an expression of the form.

$$r_1A_1 + r_2A_2 + \ldots + r_nA_n$$

where the $r_1, r_2, ..., r_n$ are numbers and $A_1, A_2, ..., A_n$ are matrices of the same size.



Exercise 1: linear combination

Write the linear system
$$\begin{cases} 3x_1 + 2x_2 - 4x_3 = 0\\ x_1 - 3x_2 + x_3 = 3 \text{ as a linear}\\ x_2 - 5x_3 = -1 \end{cases}$$
 combination of a vector.

Answer:

$$x_1 \begin{bmatrix} 3\\1\\0 \end{bmatrix} + x_2 \begin{bmatrix} 2\\-3\\1 \end{bmatrix} + x_3 \begin{bmatrix} -4\\1\\-5 \end{bmatrix} = \begin{bmatrix} 0\\3\\-1 \end{bmatrix}$$

Transpose

Given an $m \times n$ -matrix A, the **transpose** A^{\top} is the $n \times m$ -matrix which reflects along the main diagonal.

Equivalently, the transpose exchanges rows and columns.

Example

$$\begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -1 & -2 & 0 \end{bmatrix}^{\top} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ -1 & -2 \\ -3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^{\top} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Properties of the transpose

$$(\mathsf{A} + \mathsf{B})^{\top} = \mathsf{A}^{\top} + \mathsf{B}^{\top}, \quad (r\mathsf{A})^{\top} = r(\mathsf{A}^{\top}), \quad (\mathsf{A}^{\top})^{\top} = \mathsf{A}$$

Exercise 2

Write down the transpose of each of the following matrices.

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 2 & 6 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

Answer:

$$A^{T} = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}, B^{T} = \begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}, C^{T} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \text{ and } D^{T} = D.$$