Lecture 1b

Systems of linear equations (con't)

Upcoming:

- Reading HW 1b (due Thursday at 11:55pm)

How do we solve (i.e. find solutions of) a linear system?

General strategy

Use **elementary operations** to transform the system into an equivalent system whose solutions are easy to find.

We say that two systems are equivalent if they have the same set of solutions.

The three elementary operations (from least to most useful)

- Exchange two equations.
- Multiply one equation by a non-zero number.
- Add a multiple of one equation to another one.

The three elementary operations on a linear system (from least to most useful)

- Exchange two equations.
- Multiply one equation by a non-zero number.
- Add a multiple of one equation to another equation.

Exercise 2

Use elementary operations to find a solution to the following.

$$\begin{aligned} x + y + z &= 3 \\ x + y + 2z &= 4 \\ y + 2z &= 2 \end{aligned}$$

$$\begin{aligned} x + y + z &= 3 \\ x + y + 2z &= 4 \\ y + 2z &= 2 \end{aligned} \xrightarrow[\text{by } -1]{\text{multp 2nd eq}} \begin{aligned} x + y + z &= 3 \\ -x - y - 2z &= -4 \\ y + 2z &= 2 \end{aligned} \xrightarrow[\text{eq 1 + eq 2}]{\text{Replace eq 2 with}} \begin{aligned} x + y + z &= 3 \\ 0 + 0 - z &= -1 \\ y + 2z &= 2 \end{aligned}$$

- We get $\boxed{z = 1}$ from eq 2

- We substitute $z = 1$ into eq. 3 :

  $y + 2(1) = 2$

  $\boxed{y = 0}$

- We substitute $y = 0$, $z = 1$ into eq. 1 :

  $x + (0) + (1) = 3$

  $\boxed{x = 2}$

We've shown that $\boxed{(x, y, z) = (2, 0, 1)}$ is the unique solution to the linear system

$$\begin{aligned} x + y + z &= 3 \\ x + y + 2z &= 4 \\ y + 2z &= 2 . \end{aligned}$$

Let's do a sanity check !

Substitute $\underline{x = 2}$, $\underline{y = 0}$, $\underline{z = 1}$ into each of the three equations :

$$\begin{aligned} 2 + 0 + 1 &= 3 \checkmark \\ 2 + 0 + 2(1) &= 4 \checkmark \\ 0 + 2(1) &= 2 . \checkmark \end{aligned}$$

Solving linear systems: a crude strategy

- Eliminate variables by adding multiples of one equation to another (an elementary operation).
- Once you find the value of a variable, substitute it into the other equations (called back substitution).

This works pretty well for simple systems and scratch paper, but in the next video we will beef this up to an algorithm.

It also shouldn't be clear yet how this strategy deals with multiple solutions or no solutions.

Being ~~lazy~~ efficient

After each elementary operation, we only need to keep track of the coefficients and the constants.

We can do this with an **augmented matrix**.

Turning a system of linear equations into an augmented matrix

Put the numbers in a grid, put brackets around it, and add a vertical line between the coefficients and the constants.

Example

$$\begin{aligned} x + 2y + 3z &= 4 \\ 5x + 6y + 7z &= 8 \\ 9x + 10y + 11z &= 12 \end{aligned} \longmapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right]$$

Our elementary operations on linear systems become **elementary row operations** on augmented matrices.

The three elementary row operations (on an augmented matrix)

- Exchange two rows.
- Multiply one row by a non-zero number.
- Add a multiple of one row to another row.

Exercise 3

1 Convert the following linear system into an augmented matrix, 2 use elementary row operations to simplify it, 3 and determine the solutions of this system.

$$
\begin{aligned}
x + 2y + 3z &= 4 \\
5x + 6y + 7z &= 8 \\
9x + 10y + 11z &= 12
\end{aligned}
$$

Exercise 3

Step 1

linear system

$$\left.\begin{array}{l} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \\ 9x + 10y + 11z = 12 \end{array}\right\} \longrightarrow$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right]$$

Step 2 Use elementary row operation to simplify the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 9 & 10 & 11 & 12 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{array}\right]$$

Replace Row 2 with $-5\,Row_1 + Row_2$

Replace Row 3 with $-9\,Row_1 + Row_3$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- Multiply Row 2 by $-\frac{1}{4}$
- Multiply Row 3 by $-\frac{1}{8}$

Replace Row 3 with $-Row2 + Row3$

Step 3

- We can convert the last augmented matrix we wrote down to a system of linear equations

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$$

linear system

$$\begin{array}{l} x + 2y + 3z = 4 \\ 0x + y + 2z = 3 \\ \cancel{0x + 0y + 0z = 0} \end{array}$$

- The system $\begin{array}{l} x + 2y + 3z = 4 \\ y + 2z = 3 \end{array}$ is equivalent to the original system $\begin{array}{l} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \\ 9x + 10y + 11z = 12 \end{array}$

- $y = 3 - 2z$ (from eq 2)

$x = 4 - 2y - 3z$ (after sub $y = 3 - 2z$ into eq. 1)

$= 4 - 2(3 - 2z) - 3z$

$= 4 - 6 + 4z - 3z$

$= -2 + z$

- $\boxed{\text{The set of solutions is the set of triples } (-2+t, 3-2t, t) \text{ for any } t.}$

There are infinitely many sols

t is called a parameter

Recap of lecture 1a and 1b

- A general problem (finding solutions to a system of linear equations)
- Tools for simplifying that problem (elementary operations)
- A vague strategy (eliminate variables)
- A time-saving notation (augmented matrices)

Questions for next time

- When is a system of linear equations easy to solve?
- How can we decide which elementary row operation to use?
- How can we tell if we have no solutions or multiple solutions?

Wait, what are "matrices"? Aren't they a big deal?

Matrices will be an essential object in this class with lots of structure, but for now, think of augmented matrices as a notation to store numbers in a grid, like a spreadsheet.

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