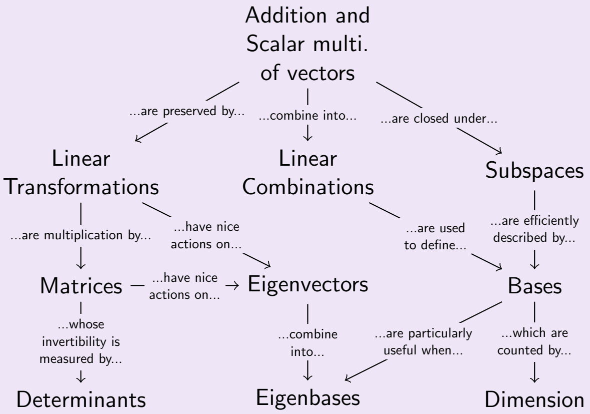
Lecture 16a

Vector Spaces

We can extend the concepts we've learned far beyond vectors.

Crucial observation #1

Most big concepts in this class can be defined in terms of addition and scalar multiplication of vectors.



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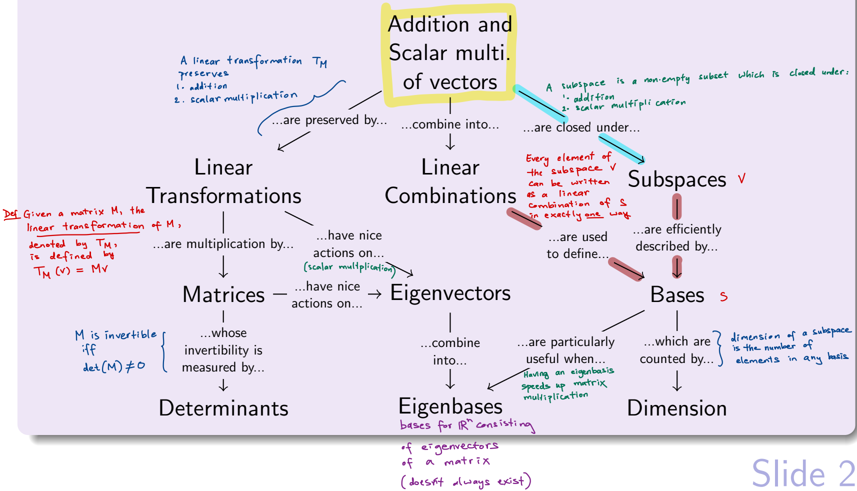

Crucial observation #2

Addition and scalar multiplication make sense for many other mathematical objects.

Examples

- Polynomials.

$$\underbrace{(1+x^2)+(7-3x+x^3)}_{\text{polynomial addition}} \qquad \underbrace{4(1+3x+4x^2)}_{\text{scalar multiplication}}$$

4 is a scalar / number

- Real-valued functions of x

function addition

$$\sin(x)+e^x \qquad \underbrace{4\ln(x)}_{\text{scalar multiplication}}$$

Goals

Generalize what we've learned so far about vectors to other kinds of objects we can add and scalar multiply.

Vector space

Definition 1: A vector space

A **vector space** is a set V in which

- there is a rule to add any two elements v, w in V, and
- there is a rule to multiply any v in V by any scalar r in $\mathbb{R}$,

such that the axioms on the next slide hold.

Intuitively, a vector space is a set of mathematical objects which collectively behave like a set of vectors.

Possibly confusing terminology

Elements of a vector space may not be vectors (as in, columns of numbers in brackets). To make this worse, some references (like our textbook) use 'vector' to refer to any element of a vector space. — I will not do this.

Axioms for vector space

Axioms (essential properties) of addition

- $u + v = v + u$ for all u, v in V. (addition is commutative)
- $(u + v) + w = u + (v + w)$ for all u, v, w in V. (addition is associative)
- There is an element 0 in V, such that for all v in V, (additive identity, called "0", exists)
$$v + 0 = 0 + v = v$$
- For each v in V, there exists $-v$ in V with (additive inverse, denoted by "$-$", exists)
$$v + (-v) = (-v) + v = 0$$

Axioms (essential properties) of scalar multiplication

- $r(u + v) = ru + rv$ for all u, v in V and any r in $\mathbb{R}$.
- $(r + s)v = rv + sv$ for all v in V and any r, s in $\mathbb{R}$.

distributivity

- $r(sv) = (rs)v$ for all v in V and any r, s in $\mathbb{R}$. (multiplication is associative)
- There is an element 1 such that $1v = v$ for all v in V. (multiplicative identity, called "1", exists)

An axiom is a fact that can't be reduced to a simpler property.

The set of vectors of height n is a vector space!

The trivial examples are the objects we are trying to generalize.

Fact 1 (The motivating examples of vector spaces)

For each positive integer n, the set $\mathbb{R}^n$ is a vector space.

We will go through previous definitions and theorems, cross out $\mathbb{R}^n$, and write 'vector space'.

$\cancel{\mathbb{R}^n}$

vector space

Fact 2 (Our first non-vector vector space)

The set of polynomials in x is a vector space, denoted $\mathbb{P}$.

Useful fact

Two polynomials are equal if and only if they have the coefficients when written in standard form: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

Exercise 1(a)

- Find a polynomial p such that $(1 + x)$ **plus** p is $x^2 + 3x + 1$.

Exercise 1(b)

- Determine whether $(x - 4)^3$ is a **scalar multiple** of $x^2 + x + 1$.

Exercise 1(c)

- Write x^2 as a **linear combination** of 1, $1 + x$, and $1 + 2x + x^2$.

Numbers like 0, 1, and 7 count as **constant** polynomials!

Exercise 1(a)

- Find a polynomial p such that $(1+x)$ **plus** p is x^2+3x+1.

$$(1+x) + p = x^2 + 3x + 1$$

$$p = x^2 + 3x + 1 - (1+x)$$

$$\boxed{p = x^2 + 2x}$$

Exercise 1(b)

- Determine whether $(x-4)^3$ is a **scalar multiple** of x^2+x+1.

Is there $\underbrace{\text{a scalar } c \text{ in } \mathbb{R}}_{\text{a number } c}$ such that

$$(x-4)^3 = c\,(x^2+x+1) \quad ?$$

Exercise 1(a)

- Find a polynomial p such that $(1+x)$ **plus** p is x^2+3x+1.

$$(1+x)+p = x^2+3x+1$$

$$p = x^2+3x+1-(1+x)$$

$$\boxed{p = x^2+2x}$$

Exercise 1(b)

- Determine whether $(x-4)^3$ is a **scalar multiple** of x^2+x+1.

Is there $\underbrace{\text{a scalar } c \text{ in } \mathbb{R}}_{\text{a number } c}$ such that

$$(x-4)^3 = c(x^2+x+1) \ ?$$

First, put the LHS in standard form:

$$(x-4)(x^2-8x+16) = c(x^2+x+1)$$

$$x^3-12x^2+(32+16)x+64 = c(x^2+x+1)$$

Since x^2+x+1 has no x^3 term, this is impossible.

$\therefore$ $(x-4)^3$ is not a scalar multiple of x^2+x+1.

Equivalently,
$(x-4)^3$ is not in the span of $\{x^2+x+1\}$, since

$(x-4)^3$ is not a linear combination of x^2+x+1.

Exercise 1(c)

- Write x^2 as a **linear combination** of 1, $1+x$, and $1+2x+x^2$.

We want to find a, b, c in $\mathbb{R}$ such that

$$x^2 = a \cdot 1 + b(1+x) + c(1+2x+x^2).$$

Note: This is a polynomial in one variable, x. The letters a, b, c are just numbers we're trying to find.

Put the RHS into standard form, so that it's easy to compare the two sides.

$$x^2 = \underline{\quad} \, x^2 + \underline{\quad} \, x + \underline{\quad} \, 1$$

Think of $1 = x^0$

We collect all terms with x^2, all terms with x, and all constant terms.

$$x^2 = \underbrace{c}\, x^2 + \underbrace{(b+2c)}\, x + \underbrace{(a+b+c)}\, 1$$

The only way the LHS equals RHS is if all coefficients match.

$$1 \cdot x^2 + 0\, x + 0 \cdot 1 = c\, x^2 + (b+2c)\, x + (a+b+c) \cdot 1$$

This tells us

$$\left.\begin{aligned} 1 &= c \\ 0 &= b + 2c \\ 0 &= a+b+c \end{aligned}\right\}$$

(a system of linear equations!)

$c = 1$

$b + 2c = 0 \Rightarrow b + 2 = 0 \Rightarrow b = -2$

$a + b + c = 0 \Rightarrow a + (-2) + 1 = 0 \Rightarrow a - 1 = 0 \Rightarrow a = 1$

$$\therefore \boxed{x^2 = \underbrace{1}_{a} \cdot (1) + \underbrace{(-2)}_{b}(1+x) + \underbrace{(1)}_{c}(1+2x+x^2)}$$

sanity check:

$$1 - 2(1+x) + 1 + 2x + x^2 \stackrel{?}{=} x^2 \checkmark$$

Definition 2: The degree of a polynomial

The **degree** of a non-zero polynomial in x is the largest power of x with non-zero coefficient.

We define $\deg(0) := -\infty$, mostly to avoid an annoying extra case.

Fact 3 (Polynomials of degree at most n)

For each positive integer n, the set of polynomials in x of degree at most n is a vector space, denoted $\mathbb{P}_n$.

Example

- $\mathbb{P}_1$ consists of polynomials $ax + b$, for a, b in $\mathbb{R}$.
- The three polynomials $(x-1)^3$, $x^2 + 3x$, and 2 are in $\mathbb{P}_3$, but the polynomials x^4 and $x^8 - 2x^3$ are not.
- $\mathbb{P}_0$ is just the constant polynomials like 0, 1, and 7, which are the same as numbers, so $\mathbb{P}_0 = \mathbb{R}$.

By a **sequence**, we mean an infinite list of real numbers.

Examples of sequences

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$	(the Fibonacci sequence)
$1, 3, 5, 7, 9, 11, 13, 15, \ldots.$	(odd numbers)
$2, 3, 5, 9, 11, 13, 17, \ldots$	(prime numbers)
$1, 3, 9, 27, 81, 243, \ldots$	(powers of 3)
$7, 12, -5, \pi, 3.5, 7, \ldots$	(Just some random numbers)

Unlike sets, order matters!

Fact 4 (The set of sequences is a vector space)

The set of sequences is a vector space, denote $\mathbb{S}$. Addition and scalar multiplication are defined term-wise.

Fact 5 (Sets of matrices of fixed size are vector spaces)

For positive integers m and n, the set of $m \times n$-matrices is a vector space, denoted $\mathbb{R}^{m \times n}$.

Addition and scalar multiplication are the matrix versions.

Definition 3: Smooth functions

A real-valued function is smooth if all higher derivatives exist.

Examples of smooth functions

$$\sin(x) \quad \cos(x) \quad e^x$$

polynomials

sums, multiples, and products of smooth functions

Fact 6 (The set of smooth functions is a vector space)

The set of smooth functions of x is a vector space, denoted $\mathcal{C}^\infty$.

This is a huge set that contains most functions you can imagine.

(Ex: Allows us to use linear algebra to study differential equations)