Lecture 15c

Eigenbases (Diagonalization)

Review Lecture 15b

Algorithm 2 (How to find an eigenbasis)

We are given an $n \times n$ matrix A.

1. Find the eigenvalues of A (by factoring the characteristic polynomial.)
2. For each eigenvalue, find a basis of the λ-eigenspace.
   - That is, a basis for $\ker(A - \lambda \mathrm{Id})$
3. Put all the vectors together into a set.
   - If there are n-many vectors, the set is an eigenbasis!
   - If there are fewer than n-many vectors, no eigenbasis exists!

Determining when a matrix has an eigenbasis without finding one

- Fact 3: A matrix A has an eigenbasis iff $\mathrm{width}(A) = \sum_\lambda \dim(E_\lambda(A))$
- Theorem 4: If an $n \times n$ matrix has n-many (distinct) eigenvalues, then it has an eigenbasis.
  (If the matrix has fewer than n eigenvalues, we have to do more work.)

The idea that eigenbases 'simplify multiplication' can be encoded into a special factorization of A, called a diagonalization of A.

Let's start with a computation!

Suppose $S := \{v_1, v_2, \ldots, v_n\}$ is an eigenbasis for an $n \times n$ matrix A. By def, S is a basis for $\mathbb{R}^n$.

Let $P := \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$, concatenation of S. Then P is invertible (since S is a basis for $\mathbb{R}^n$).

Then $AP = \begin{bmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_n \\ | & | & & | \end{bmatrix}$

This is true in general.
The column vectors of a product BC are

$B(\text{1st col of } C)$, $B(\text{2nd col of } C)$, $\ldots$, $B(\text{last col of } C)$.

$= \begin{bmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \\ | & | & & | \end{bmatrix}$

where λ_1 is the eigenvalue for v_1,
λ_2 is the eigenvalue for v_2,
$\ldots$
λ_n is the eigenvalue for v_n.

[Note: $\lambda_1, \lambda_2, \ldots, \lambda_n$ don't have to be distinct]

$= \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \overbrace{\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}}$

Call this diagonal matrix D

So $AP = PD$, so $\boxed{A = PDP^{-1}}$.

From above, we have $AP = PD$. Since P is invertible, we can rewrite $AP = PD$ as ... $A = PDP^{-1}$

Theorem 5 (Diagonalizing a matrix with an eigenbasis)

Let $v_1, v_2, \ldots, v_n$ be an eigenbasis for A, and let λ_i denote the eigenvalue of v_i. Then we can factor A as

$$A = PDP^{-1}$$

where

- P is the concatenation of the eigenbasis.
- D is the diagonal matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the diagonal.

Such a factorization is called a **diagonalization** of A.

Thus, a matrix with an eigenbasis is called **diagonalizable**.

Example:

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{-2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}}_{P^{-1}}$$

$$\underbrace{\begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{P^{-1}}$$

Both P and D depend on an ordering of the eigenvalues and the eigenvectors, and both have to be in the same order.

Exercise 8

Diagonalize the following matrix; that is, write it as PDP^{-1} for some diagonal matrix D.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Exercise 8 solution (pg 1/2)

Diagonalize the following matrix; that is, write it as PDP^{-1} for some diagonal matrix D.

$$A := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

First, use Algorithm 2 to find an eigenbasis for A.

Step ①: Find eigenvalues of A

- Characteristic polynomial of A is

$$\begin{aligned} p_A(x) &= \det(x\,\mathrm{Id} - A) \\ &= \det\left(\begin{bmatrix} x-2 & -1 \\ -1 & x-2 \end{bmatrix}\right) \\ &= (x-2)(x-2) - 1 \\ &= x^2 - 4x + 4 - 1 \\ &= x^2 - 4x + 3 \\ &= (x-1)(x-3) \end{aligned}$$

- Roots of $p_A(x)$ are $\lambda_1 = 1$, $\lambda_2 = 3$.
  So the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 3$.

Step ②: Find a basis for each eigenspace of A

- Find a basis for the 1-eigenspace of A,
  $E_1(A) = \ker(A - 1\,\mathrm{Id})$

$$\left[\begin{array}{cc|c} 2-1 & 1 & 0 \\ 1 & 2-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto -R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(Col 2 has no leading 1)
Let $y = t$
$x + y = 0 \Rightarrow x = -t$

General solution: $\begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

A basis for $E_1(A)$ is $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

- Find a basis for the 3-eigenspace of A,
  $E_3(A) = \ker(A - 3\,\mathrm{Id})$

$$\left[\begin{array}{cc|c} 2-3 & 1 & 0 \\ 1 & 2-3 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto R_1 + R_2]{} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \mapsto -R_1]{} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

(Col 2 has no leading 1)
Let $y = t$
$x - y = 0 \Rightarrow x = t$

General solution: $\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

A basis for $E_3(A)$ is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Step ③: Put all basis vectors together

$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ has $n=2$ vectors (not fewer than two),

so it is an eigenbasis for A.

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Exercise 8 (solution pg 2/2)

Diagonalize the following matrix; that is, write it as PDP^{-1} for some diagonal matrix D.

$$A := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

From the previous pages, we found an eigenbasis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ of A

with corresponding eigenvalues $\lambda_1 = 1$ $\lambda_2 = 3$

To diagonalize A, we need P, $\underbrace{D}_{\text{diagonal matrix}}$, P^{-1} where $A = PDP^{-1}$

$P =$ concatenation of the vectors in the eigenbasis $= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{-1.1-1.1} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$\left(\text{if } P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

$D =$ diagonal matrix with eigenvalues on the diagonal (in the same order as the concatenation P of the eigenvectors) $= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$$A = P D P^{-1}$$

$$\boxed{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}}$$

Sanity check: Confirm that PDP^{-1} equals A.

— the end —

An application of Theorem 5 (Rapid exponents via diagonalization)

Let A be a matrix such that

$$A = BDB^{-1}$$

for some diagonal matrix D. Then, for all $n \geq 0$,

$$A^n = BD^nB^{-1}$$

The entries of D^n are the nth power of the entries of D.

E.g. $A^2 = (BDB^{-1})(BDB^{-1})$
$= BDDB^{-1}$
$= BD^2B^{-1}$

$A^{-1} = (BDB^{-1})^{-1}$
$= BD^{-1}B^{-1}$

Exercise 9

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Compute A^2 using the factorization.
(b) Compute A^{-1}.
(c) Compute A^{100}.

Exercise 9 (solution)

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{B^{-1}}$$

(a) Compute A^2 using the factorization.

$A = BDB^{-1}$

$A^2 = BDB^{-1}BDB^{-1}$

$\quad = BDDB^{-1}$

Note: $D^2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 1^2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Compute A^{-1}.

$A = BDB^{-1}$

$A^{-1} = BD^{-1}B^{-1}$

Note: $D^{-1} = \begin{bmatrix} 3^{-1} & 0 & 0 \\ 0 & 2^{-1} & 0 \\ 0 & 0 & 1^{-1} \end{bmatrix}$

$A^{-1} = B\begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}B^{-1}$

(c) Compute A^{100}.

$A = BDB^{-1}$

$A^{100} = \underbrace{(BDB^{-1})(BDB^{-1})\ldots(BDB^{-1})}_{100 \text{ times}}$

$= B\underbrace{DD\ldots D}_{100 \text{ times}}B^{-1}$

$= BD^{100}B^{-1}$

Note: $D^{100} = \begin{bmatrix} 3^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 1^{100} \end{bmatrix}$

$A^{100} = B\begin{bmatrix} 3^{100} & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 1^{100} \end{bmatrix}B^{-1}$

3^{100} is much bigger than 2^{100} and 1,

so A^{100} is close to $B\begin{bmatrix} 3^{100} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}B^{-1}$.

— the end —

One of the most unexpected and powerful results in linear algebra is that an important class of matrices always has a nice eigenbasis.

Def: A set of vectors is called **orthogonal** if the dot product of any pair of vectors is 0. (Note: Two vectors are orthogonal if and only if they are perpendicular to each other.)

Theorem 6 (The Spectral Theorem)

If A is a symmetric matrix, then A has a basis of **orthogonal** eigenvectors.

equal to its transpose

In fact, this goes both ways. If a matrix has an orthogonal eigenbasis, then the matrix must be symmetric.

This theorem, (and its generalizations) are of fundamental importance in differential equations, statistics, acoustics, quantum mechanics, data science, and countless other fields.

Exercise 10

Verify that the eigenbasis we found in Exercise 8 was orthogonal

Solution

From Exercise 8: $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is an eigenbasis for $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Dot product of the two vectors: $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 \cdot 1 + 1 \cdot 1 = 0$

$\therefore$ $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal eigenbasis.

Exercise 11

Is the following matrix diagonalizable?

$$\begin{bmatrix} 100 & 3 & \sqrt{2} & \pi \\ 3 & -70 & e^{16} & 0 \\ \sqrt{2} & e^{16} & 9 & \sqrt{2}^{\sqrt{2}} \\ \pi & 0 & \sqrt{2}^{\sqrt{2}} & 3^{3^{3}} \end{bmatrix}$$

Exercise 11

Is the following matrix diagonalizable?

$$M := \begin{bmatrix} 100 & 3 & \sqrt{2} & \pi \\ 3 & -70 & e^{16} & 0 \\ \sqrt{2} & e^{16} & 9 & \sqrt{2}^{\sqrt{2}} \\ \pi & 0 & \sqrt{2}^{\sqrt{2}} & 3^{3^3} \end{bmatrix}$$

Solution

- M is symmetric $(M^T = M)$, so by the Spectral Thm M has an (orthogonal) eigenbasis.
- Since M has an eigenbasis, it is diagonalizable by Thm 5.