Lecture 15a

Eigenbases

Review

Recap: Finding a basis for standard subspaces

Subspace	**A method to find one basis**
Image of A	Put A in REF, keep columns of A corresponding to L1s
Span of $\{v_1, ...v_n\}$	$= \mathrm{im}(\text{concatenation})$, use $\uparrow$
Kernel of A	Put A into REF, find gen. sol. to $Ax = 0$ rewrite as linear combination, keep vectors
Solutions to HSLE	$= \ker(\text{coeff. matrix})$, use $\uparrow$
λ-eigenspace of A	$= \ker(A - \lambda \mathrm{Id})$, use $\uparrow$

Review

Recap: Finding the dimension of standard subspaces

Subspace	**Dimension**
Image of A	$\mathrm{rank}(A)$
Span of $\{v_1, ... v_n\}$	$\mathrm{rank}(\text{concatenation})$
Kernel of A	$\mathrm{width}(A) - \mathrm{rank}(A)$
Solutions to HSLE	$(\#\text{ of variables}) - \mathrm{rank}(\text{coeff. matrix})$
λ-eigenspace of A	$\mathrm{width}(A) - \mathrm{rank}(A - \lambda \mathrm{Id})$

$\ker(A - \lambda \mathrm{Id})$

In each case, the dimension is easy if we know a certain rank.

Exercise 4 (Review from Lecture 14a)

(a) Find a basis of the 2-eigenspace of

$$M := \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

(b) What is the dimension of this eigenspace?

(Recall) Def The λ-eigenspace of M is $\{ v \text{ in } \mathbb{R}^{\text{width}(M)} \text{ where } Mv = \lambda v \}$

So the 2-eigenspace of M is $W := \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 \text{ where } \underbrace{M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{(M - 2\,\mathrm{Id}_{3\times 3}) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \right\}$

Note $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ where } \begin{bmatrix} 2-2 & 2 & 4 \\ 0 & 1-2 & -2 \\ 0 & 1 & 4-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

i.e. $W = \ker \left(\begin{bmatrix} 0 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} \right).$

(Review from Exercise 4 Lecture 14a)

Algorithm 1 (Find a basis for the kernel of a matrix) says we just need to solve for $\begin{bmatrix} 0 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

write the solutions as linear combinations of a set S of vectors — the set S will be a basis for W.

Row reduce:

$$\left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto -R_1 + R_3]{R_2 \mapsto R_1 + R_2} \overbrace{\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}^{\text{REF}}$$

Let $\boxed{x = t}$
Let $\boxed{z = r}$

1st and 3rd columns have no leading 1

Note: If there had been k number of columns without a leading 1, $\dim(W) = k$

Back substitution:
$y + 2z = 0 \Rightarrow y + 2r = 0 \Rightarrow y = \boxed{-2r}$

General solution: $\begin{bmatrix} t \\ -2r \\ r \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2r \\ r \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.

a) $\therefore$ $\boxed{\text{A basis for } W \text{ is } \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right\}}$.

b) So $\dim(W) = 2$.

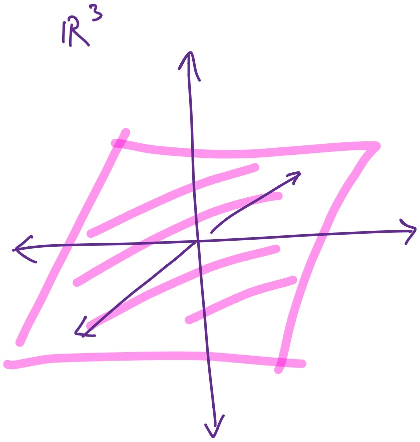
Let's do a sanity check. Check that at least $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a subset of the 2-eigenspace of M.

Check: $M\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \stackrel{?}{=} 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $M\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \checkmark$$

So at least our set of two vectors is a subset of W.



The 2-eigenspace of M, W, has dimension 2. So W is a plane where every vector in W is stretched by a factor of 2 when multiplied by M.

Motivation

Recall: Eigenvectors turn matrix multiplication into scalar multiplication.
$A v = \lambda v$

Simplifying multiplication by A

Let A be a specific matrix. We can simplify multiplication by A (that is, the action of the linear transformation T_{A}).

- The action on any vector can be reduced to the action on a basis. Specifically, if $\mathrm{w} = c_1\mathrm{v}_1 + c_2\mathrm{v}_2 + \cdots + c_n\mathrm{v}_n$, then

  If we know what A does to each basis vector, we can compute

  $$\mathrm{Aw} = c_1\mathrm{Av}_1 + c_2\mathrm{Av}_2 + \cdots + c_n\mathrm{Av}_n$$

  the action of A on any vector in the subspace

- Matrices act on their eigenvectors in a particularly simple way.

  $$\mathrm{Av} = \lambda\mathrm{v}$$

So, we can simplify a matrix multiplication using a basis of eigenvectors.

We call 'a basis of eigenvectors' an **eigenbasis**.

Definition 1: Eigenbases

An **eigenbasis** for an $n \times n$-matrix A is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

1 2

Exercise 1

$$\text{Let} \quad S := \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad A := \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

Do the vectors of S form an eigenbasis for A?

Check : 1 Do the vectors form a basis for $\mathbb{R}^3$?

2 Is each vector an eigenvector of A ?

Check 2 first.

The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$ are eigenvectors (with eigenvalue 2) of A (by the previous exercise).

$A\begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}2 & 2 & 4\\0 & 1 & -2\\0 & 1 & 4\end{bmatrix}\begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}2\cdot 2 - 2\cdot 1 + 4\\ -1-2\\ -1+4\end{bmatrix} = \begin{bmatrix}6\\-3\\3\end{bmatrix} = 3\begin{bmatrix}2\\-1\\1\end{bmatrix}$, so $\begin{bmatrix}2\\-1\\1\end{bmatrix}$ is an eigenvector of A (with eigenvalue 3)

$\therefore$ Yes, each vector of S is an eigenvector of A.

① Do the vectors form a basis for $\mathbb{R}^3$?

Recall: To check that a set of n-many vectors is a basis for $\mathbb{R}^n$, we just need to check that the concatenation of the vectors

is invertible

OR

has rank n

OR

has determinant nonzero. I will check this

$S := \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\end{bmatrix} \right\}$ Concatenation $C := \begin{bmatrix}1 & 0 & 2\\0 & -2 & -1\\0 & 1 & 1\end{bmatrix}$

$\det\begin{bmatrix}1 & 0 & 2\\0 & -2 & -1\\0 & 1 & 1\end{bmatrix} = -\det\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & -2 & -1\end{bmatrix} = -\det\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} = -1 \neq 0$

swapping two rows multiply the determinant by -1

$R_3 \mapsto 2R_2 + R_3$ (Adding a multiple of one row to another row does not change the determinant)

So the vectors of S form a basis for $\mathbb{R}^3$.

So they form an eigenbasis for A.

— the end —

Observation 1 (Matrix multiplication and eigenbases)

Let $v_1, v_2, \ldots, v_n$ be an eigenbasis for A, and let λ_i denote the eigenvalue of v_i. If $w = c_1v_1 + c_2v_2 + \cdots + c_nv_n$, then

$$\begin{aligned} Aw &= c_1\lambda_1 v_1 + c_2\lambda_2 v_2 + \cdots + c_n\lambda_n v_n \\ A^2w &= c_1\lambda_1^2 v_1 + c_2\lambda_2^2 v_2 + \cdots + c_n\lambda_n^2 v_n \\ A^mw &= c_1\lambda_1^m v_1 + c_2\lambda_2^m v_2 + \cdots + c_n\lambda_n^m v_n \end{aligned}$$

This reduces matrix multiplication to several scalar multiplications!

Exercise 2

Compute

$$\left(\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}\right)^{100} \overbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}^{\text{Call this } w}$$

From previous exercise, the matrix $A := \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$ has an eigenbasis $S := \left\{ \underbrace{\overset{v_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}}}_{\text{eigenvalue } 2}, \underbrace{\overset{v_3}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}}_{\text{eigenvalue } 3} \right\}$

• Write $w = \begin{bmatrix}1\\1\\1\end{bmatrix}$ as a linear combination of our eigenbasis

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\-2\\1\end{bmatrix} + c_3 \begin{bmatrix}2\\-1\\1\end{bmatrix}$$

Solve $$\underbrace{\begin{bmatrix}1 & 0 & 2\\0 & -2 & -1\\0 & 1 & 1\end{bmatrix}}_{\text{concatenation of our basis vectors}} \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Row reduce

$$\left[\begin{array}{ccc|c}1 & 0 & 2 & 1\\0 & -2 & -1 & 1\\0 & 1 & 1 & 1\end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c}1 & 0 & 2 & 1\\0 & 1 & 1 & 1\\0 & -2 & -1 & 1\end{array}\right] \xrightarrow[R_3 \mapsto 2R_2 + R_3]{} \left[\begin{array}{ccc|c}1 & 0 & 2 & 1\\0 & 1 & 1 & 1\\0 & 0 & 1 & 3\end{array}\right]$$

Note: Every column left of the vertical line has a leading 1, so we have one unique solution (as expected, since S is a basis for $\mathbb{R}^3$)

Back substitute:

$$c_1 + 2c_3 = 1 \Rightarrow c_1 + 2(3) = 1 \Rightarrow \boxed{c_1 = 1 - 6 = -5}$$

$$c_2 + c_3 = 1 \Rightarrow c_2 + 3 = 1 \Rightarrow \boxed{c_2 = -2}$$

$$\boxed{c_3 = 3}$$

$$\therefore \begin{bmatrix}1\\1\\1\end{bmatrix} = -5\begin{bmatrix}1\\0\\0\end{bmatrix} - 2\begin{bmatrix}0\\-2\\1\end{bmatrix} + 3\begin{bmatrix}2\\-1\\1\end{bmatrix}.$$

sanity check (at least one of the entries):

$$1 \overset{?}{=} -5(1) - 2 \cdot 0 + 3(2) = -5 + 6 = 1 \ \checkmark$$

$$w = -5 v_1 - 2 v_2 + 3 v_3$$

By Observation 1, we have

$$A^{100} w = -5 A^{100} v_1 - 2 A^{100} v_2 + 3 A^{100} v_3$$

$$= -5 \cdot \underbrace{2^{100} v_1}_{\text{because } A\begin{bmatrix}1\\0\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix}} - 2 \cdot \underbrace{2^{100} v_2}_{\text{because } A\begin{bmatrix}0\\-2\\1\end{bmatrix} = 2\begin{bmatrix}0\\-2\\1\end{bmatrix}} + 3 \cdot \underbrace{3^{100} v_3}_{\text{because } A\begin{bmatrix}2\\-1\\1\end{bmatrix} = 3\begin{bmatrix}2\\-1\\1\end{bmatrix}}$$

$$\boxed{\text{So } A^{100}\begin{bmatrix}1\\1\\1\end{bmatrix} = -5 \cdot 2^{100}\begin{bmatrix}1\\0\\0\end{bmatrix} - 2 \cdot 2^{100}\begin{bmatrix}0\\-2\\1\end{bmatrix} + 3 \cdot 3^{100}\begin{bmatrix}2\\-1\\1\end{bmatrix}}$$

Warning!

Eigenbases don't always exist!

Exercise 3

Show that the following matrix does not have an eigenbasis.

$$B := \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

- To find eigenvalues, write down the characteristic polynomial of B:

  $$p_B(x) = \det(x\,\mathrm{Id} - B) = \det\begin{bmatrix} x-2 & -3 \\ 0 & x-2 \end{bmatrix} = (x-2)(x-2).$$

- Find the roots of $p_B(x)$:

  $x=2$ is the only root

  So the only eigenvalue of B is $\lambda = 2$.

• Find the 2-eigenspace of B.

(Recall: the 2-eigenspace of B is $\ker(B - 2\,Id)$

$$= \ker\left(\begin{bmatrix} 2-2 & 3 \\ 0 & 2-2 \end{bmatrix}\right)$$

$$= \ker\left(\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\right) \;)$$

Row reduce to find solution to $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \mapsto \frac{1}{3}R_1]{} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $x = t$ (since 1st col has no leading 1)

$y = 0$

General solution of $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So the 2-eigenspace of B consists of vectors of the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$.

Two vectors of the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$ cannot be linearly independent because they are multiple of each other.

So no eigenbasis exists for B.

— the end —