Lecture 14b

Rank and Dimension (No row reduce)

Review

So far, we have two main algorithms for finding bases and the dimension of special subspaces.

Finding a basis and dimension for standard subspaces

Lec 13

Subspace	Method to find one basis	Dimension
1. Image of A	Columns with L1 in REF	rank
2. Span of $\{v_1, \ldots v_n\}$	$=$ im(concatenation), use $\uparrow$	$\uparrow$
3. Kernel of A	Vectors in general solution	width $-$ rank
4. Solutions to HSLE	$=$ ker(coeff. matrix), use $\uparrow$	$\uparrow$
5. λ-eigenspace of A	$= \ker(A - \lambda \mathrm{Id})$, use $\uparrow$	$\uparrow$

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In each case, the dimension is easy if we know a certain rank.

Goal

Compute rank and dimension without row reduction

For a fixed matrix A, we have two simple formulas.

$$\dim(\mathrm{im}(A)) = \mathrm{rank}(A)$$
$$\dim(\ker(A)) = \mathrm{width}(A) - \mathrm{rank}(A)$$

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Each formula requires the rank of A...but their sum does not.

The Rank-Nullity Theorem

Let A be any matrix. Then

$$\underbrace{\dim(\mathrm{im}(A))}_{\mathrm{rank}(A)} + \underbrace{\dim(\ker(A))}_{\text{'nullity' of A}} = \mathrm{width}(A)$$

Nullity is an archaic word for the dimension of the kernel.

Exercise 5

The image of the following matrix is a plane in $\mathbb{R}^3$.

$$A := \begin{bmatrix} 1 & -1 & 4 \\ 2 & 0 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

Find the dimension of the kernel of A.

$\text{im}(A)$ is a plane, so $\dim(\text{im}(A)) = 2$.

Recall that $\dim(\text{im}(A))$ could be found by row-reducing and counting the number of columns with leading 1, so (in general) $\dim(\text{im}(A)) = \text{rank}(A)$. So $\text{rank}(A) = 2$

We've seen (in general) $\dim(\ker(A)) = \text{width}(A) - \text{rank}(A)$
$= 3 - 2$
$= \boxed{1}$

Side note: Geometrically, this means that the set of vectors sent to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ by T_A is a line through the origin in 3D space.

Alternatively: Use the Rank-Nullity Thm
$\dim(\ker(A)) + \dim(\text{im}(A)) = \text{width}(A)$
So $\dim(\ker(A)) = \text{width}(A) - \dim(\text{im}(A))$
$= 3 - 2 = 1.$

Subspaces with dimension 0

What dimensions are possible, and what do they tell us?
Let's consider several special cases.

Definition: The zero subspace

The **zero subspace** of $\mathbb{R}^n$ only contains the zero vector.

Note: this is the only subspace with finitely many elements.

- If there are more than just one element in a subspace W, there must be infinitely many elements in W.

Fact/Definition (The subspace of dimension 0)

The only 0 dimensional subspace of $\mathbb{R}^n$ is the zero subspace.

Fact (Matrices of rank 0)

The only $m \times n$ matrix of rank 0 is the zero matrix.

- If an REF matrix has no leading 1, it's the zero matrix
- A nonzero matrix cannot be turned into the zero matrix using elementary row operations.

Subspaces with maximum dimension

We can also consider the case of maximum dimension.

Recall Fact: A bound on dimension of subspaces

A subspace of $\mathbb{R}^n$ has dimension at most n.
(A basis for a subspace of $\mathbb{R}^n$ has n or fewer vectors)

Fact (The subspace of maximum dimension)

The only n dimensional subspace of $\mathbb{R}^n$ is all of $\mathbb{R}^n$.

For example, if a subspace W of $\mathbb{R}^5$ has dimension 5, then W must be the entire $\mathbb{R}^5$ ($W = \mathbb{R}^5$).

Matrices of maximal rank aren't unique, e.g. any invertible matrix has largest possible rank.

Subspaces with dimension 1

Fact (Subspaces of dimension 1)

A subspace is 1 dimensional if it consists of multiples of a non-zero vector

(Geometrically, a line through the origin in n dimensional space)

Fact (Matrices of rank 1)

A non-zero matrix has rank 1 if and only if all the columns are multiples of each other.

(Equivalently, a non-zero matrix has rank 1 iff all the rows are multiples of each other)

Example

three times R1

-1 times R1

$$\text{rank}\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ -1 & -2 & -1 \end{bmatrix} = 1$$

If we were to do row reduction, we would get $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ -1 & -2 & -1 \end{bmatrix} \dashrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Exercise 6

$$\text{Let} \quad A := \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

(a) Find the rank of the following matrix. (b) What is $\dim(\mathrm{im}(A))$?
(c) What is the dimension of the kernel of A?

[Let's practice applying the methods from this lecture — no row reduce]

(a) Possible rank for a 2x2 matrix:

$\cancel{0}$ — A is not the zero matrix

$\cancel{1}$ — The rows of A are not multiples of each other

2

$\boxed{\mathrm{rank}(A) = 2}$

(b) $\dim(\mathrm{im}(A)) = \underbrace{\mathrm{rank}(A)}_{} = \boxed{2}$

\# of columns of REF with leading 1

(c) $\dim(\ker(A)) = \underbrace{\mathrm{width}(A) - \mathrm{rank}(A)}_{}$

Think/remember: \# of columns of REF with no leading 1

$= 2 - 2 = \boxed{0}$

Note: This means $\ker(A)$ is the zero subspace, $\ker(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

So the only vector sent to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ by T_A is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Exercise 7

$$\text{Let} \quad B := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

(a) Find the rank of B.
(b) What is the dimension of the image of B?
(c) What is the dimension of the kernel of B?

(a) Possible ranks: $\cancel{0}$ $\cancel{1}$ 2 $\cancel{3}$

B is not the zero matrix

The 1st and 2nd columns are multiple of each other, but the 3rd column is not a multiple of the 1st col.

The 1st and 2nd columns are equal. Recall this means $\det(B) = 0$. So B is not invertible. So $\operatorname{rank}(B)$ is not maximum.

So $\boxed{\operatorname{rank}(B) = 2}$

(b) $\dim(\operatorname{Im}(B)) = \operatorname{rank}(B) = \boxed{2}$

(c) $\dim(\ker(B)) = \operatorname{width}(B) - \operatorname{rank}(B)$
$= 3 - 2 = \boxed{1}$

This means $\ker(B)$ is a line through the origin in 3D space.

We can also relate dimension to containment between subspaces.

Theorem 3: Subspaces contained in other subspaces

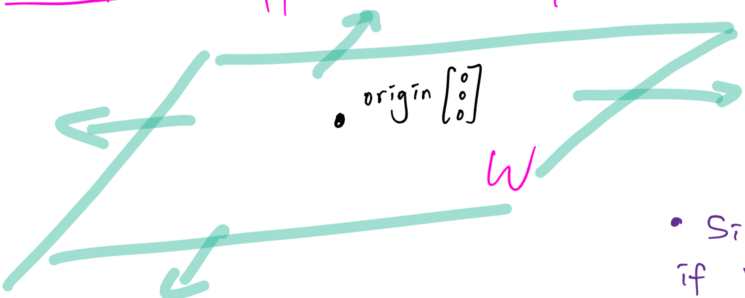
Let V and W be subspaces of $\mathbb{R}^n$.

(a) • If V is contained in W, then $\dim(V) \leq \dim(W)$.

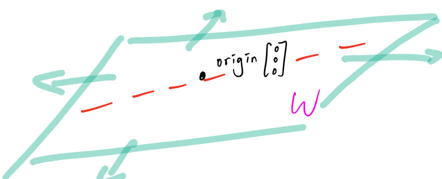
(b) • If V is contained in W and $\dim(V) = \dim(W)$, then $V = W$.

Think of dimension as a measure of how "big" a subspace is.

Example Suppose W is a plane through the origin in $\mathbb{R}^3$



• If V is a subspace contained in W with dimension 2, then V must be the entire plane W. (by (b))



• Since W has dimension 2, if V is contained in W, then V can have dimension 0, 1, or 2 (by (a)).

– If V has dimension 1, then V must be some line in W through the origin.

– If V has dimension 0, then $V = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix} \right\}$, the zero subspace of $\mathbb{R}^3$.

Exercise 8

Let A and B be two matrices such that the product AB is defined.

1. Show that $\mathrm{im}(\mathsf{AB})$ is contained in $\mathrm{im}(\mathsf{A})$.
2. Show that $\mathrm{rank}(\mathsf{AB}) \leq \mathrm{rank}(\mathsf{A})$.
3. Show that $\mathrm{rank}(\mathsf{AB}) \leq \mathrm{rank}(\mathsf{B})$. **Hint: Is there a trick to reverse the order of multiplication without changing the rank?** I'll use transpose, since $\mathrm{rank}(M)=\mathrm{rank}(M^T)$ and $(CD)^T=D^TC^T$

This shows the following general principle, which is virtually impossible to show from the leading 1s definition.

Rank and matrix multiplication

$$\mathrm{rank}(\mathsf{AB}) \leq \min(\mathrm{rank}(\mathsf{A}), \mathrm{rank}(\mathsf{B}))$$

Exercise 8

Let A and B be two matrices such that the product AB is defined.

1. Show that $\mathrm{im}(AB)$ is contained in $\mathrm{im}(A)$.

We want to show $\mathrm{im}(AB) \subset \mathrm{im}(A)$

["$\subset$" means: "contained in" or "is a subset of"]

Let v be in $\mathrm{im}(AB)$; that is,

there is some w such that

$$v = (AB)w.$$
$$= A(\underbrace{Bw}_{\text{(a vector)}})$$

Therefore, v is in $\mathrm{Im}(A)$.

Since this computation works for all v in $\mathrm{im}(AB)$,

we can say that $\mathrm{im}(AB)$ is in $\mathrm{im}(A)$. — the end of ① —

Exercise 8

Let A and B be two matrices such that the product AB is defined.

1. Show that $\mathrm{im}(AB)$ is contained in $\mathrm{im}(A)$.
2. Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$. Will show (2)

Since $\mathrm{im}(AB)$ is contained in $\mathrm{im}(A)$ by part (1),

Theorem 3 tells us that

$$\underbrace{\dim(\mathrm{im}(AB))}_{\mathrm{rank}(AB)} \leq \underbrace{\dim(\mathrm{im}(A))}_{\mathrm{rank}(A)},$$
$$\mathrm{rank}(AB) \leq \mathrm{rank}(A).$$

— the end of (2) —

Exercise 8

Let A and B be two matrices such that the product AB is defined.

1. Show that $\mathrm{im}(AB)$ is contained in $\mathrm{im}(A)$.
2. Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.
3. Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$. **Hint: Is there a trick to reverse the order of multiplication without changing the rank?** I'll use transpose, since $\mathrm{rank}(M) = \mathrm{rank}(M^T)$ and $(CD)^T = D^T C^T$

Recall:

a. Rank of a matrix equals rank of its transpose

$$\mathrm{rank}(M) = \mathrm{rank}(M^T)$$

b. $(CD)^T = D^T C^T$.

$$\mathrm{rank}(AB) \overset{(a)}{=} \mathrm{rank}((AB)^T)$$
$$\overset{(b)}{=} \mathrm{rank}(B^T A^T)$$
$$\leq \mathrm{rank}(B^T) \quad \text{by part (2)}$$
$$\overset{(a)}{=} \mathrm{rank}(B)$$

So $\boxed{\mathrm{rank}(AB) \leq \mathrm{rank}(B).}$