Lecture 14a

Basis Algorithm~~s~~ for the kernel of a matrix

Review

Definition (Subspaces of $\mathbb{R}^n$)

A **subspace** of $\mathbb{R}^n$ is a ⓪ non-empty subset of $\mathbb{R}^n$ which ① is closed under addition and ② scalar multiplication.

- Subspaces generalize linear objects like lines and planes (to higher dimension)
- Many sets we've studied are subspaces: solution sets to homogeneous linear systems, eigenspaces, kernels, images, spans.

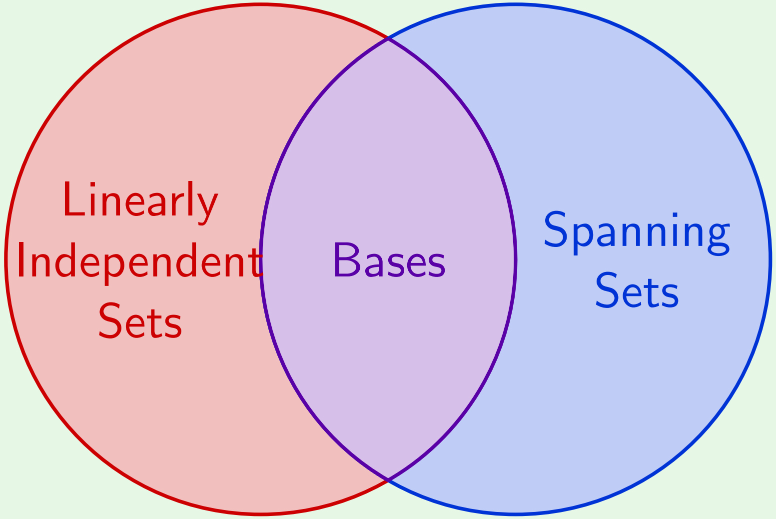
Definition (Bases for a subspaces)

A **basis** S of a subspace V is a list of vectors such that every vector in V can be written uniquely as a linear combination of the vectors in S.

- Useful for efficiently encoding subspaces in a finite list of vectors.
- Gives us the notion of dimension

Review

Special sets in a subspace



Review

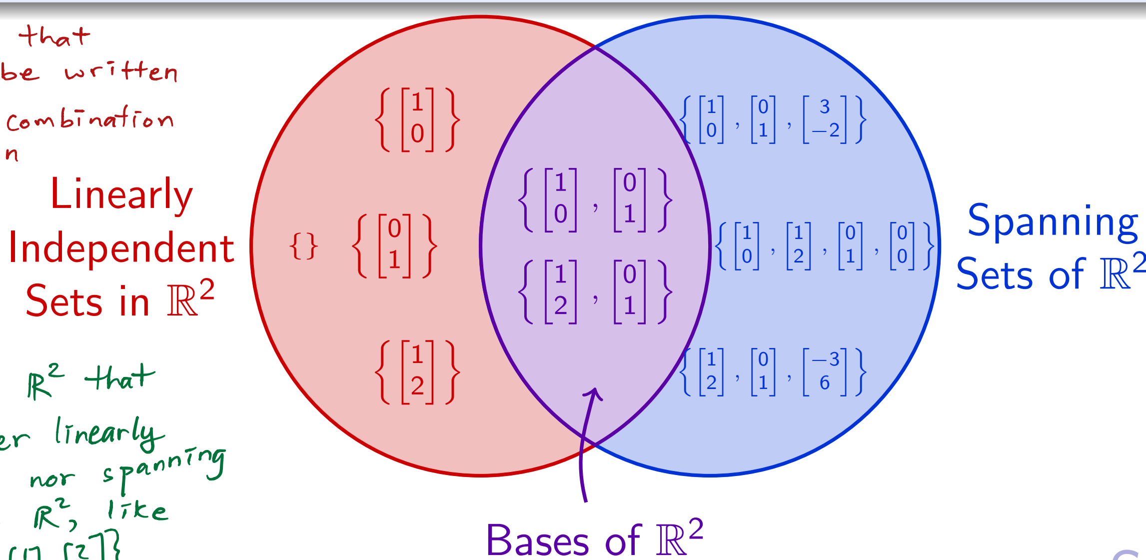
Recall: Getting a basis from the other special sets

- Every spanning set contains a basis (usually several).

  a set which is big enough to span the subspace, but it could be too big

  - One way (Algorithm 5 from the last lecture): Keep the columns with a leading 1 in the REF.
- Every linearly independent set can be extended to a basis.
  - No nice algorithm for this.

small enough that nothing can be written as a linear combination in more than one way



Subsets of $\mathbb{R}^2$ that are neither linearly independent nor spanning $\mathbb{R}^2$, like sets for $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$

Review

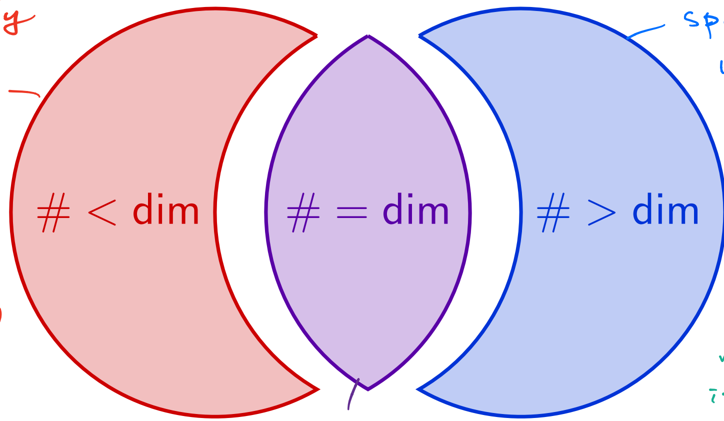
Recall: Dimension and number of vectors

- Every basis for a V contains exactly $\dim(V)$-many vectors.

Theorem 7 from the last lecture:

- Every spanning set for V contains at least $\dim(V)$-many vectors, and equality implies it is a basis.
- Every linearly independent set in V contains at most $\dim(V)$-many vectors, and equality implies it is a basis.

If I have a linearly independent subset of V which is not a basis, the number of vectors is strictly smaller than $\dim(V)$



spanning sets of V which are not bases of V have strictly more vectors than $\dim(V)$.

Note: If a subset of V is neither linearly indep. nor a spanning set of V, it can have any number of vectors.

a basis of V has exactly $\dim(V)$ vectors

Review

Recall Theorem 7 from the last lecture (The '2 out of 3' rule)

To show a set of vectors in a subspace V is a basis, you only need to check 2 of the following 3:

- The set is linearly independent.
- The set is a spanning set for V.
- The number of vectors in the set equals $\dim(V)$.

Exercise 1 (Review Exercise 10 from live class lecture on Lec 13b)

Exercise 10 Suppose W is a subspace and we know $\dim(W) = 4$ (Note: this means any basis for W has $\underline{4}$ vectors).

Let $T := \left\{ \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a subset of W.

Check whether T is a basis for W.

Solution

> **Theorem 7, restated (The '2 out of 3 Rule' for checking a basis)**
>
> Let $v_1, v_2, \ldots, v_k$ be elements in V. If any 2 of the following 3 properties are true, then the 3rd one is automatically true.
>
> - $v_1, v_2, \ldots, v_k$ is a spanning set.
> - $v_1, v_2, \ldots, v_k$ is linearly independent.
> - The dimension of V is k.
>
> So, if any 2 of these are true, then $v_1, v_2, \ldots, v_k$ is a basis for V.

- Check that the number of vectors in T is $\dim(W)$. ✓
- Check that T is a linearly independent set.

That is, check that

$$x \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + z \begin{bmatrix} 6 \\ 7 \\ 8 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has one unique solution, the trivial solution $x=0$, $y=0$, $z=0$, $w=0$.

$$\begin{bmatrix} 6 & 1 & 6 & 0 \\ 7 & 2 & 7 & 1 \\ 8 & 3 & 8 & 1 \\ 9 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 6 & 1 & 6 & 0 & 0 \\ 7 & 2 & 7 & 1 & 0 \\ 8 & 3 & 8 & 1 & 0 \\ 9 & 4 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{cccc|c} 1 & * & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The REF having a leading 1 in every column to the left of the vertical line means the system has one unique solution.

Hence T is linearly independent.

$$\boxed{\text{Since } \dim(W) \text{ is equal to the number of vectors in } T \text{ and } T \text{ is linearly independent, the set } T \text{ is a basis for } W.}$$

— the end —

Next up

Computing a basis for the kernel of A (i.e. solutions to $Av = 0$).

Exercise 2

a. Find the general solution to the following matrix equation.

$$\underbrace{\begin{bmatrix} 2 & -2 & -4 & 4 \\ -1 & 1 & 3 & 2 \end{bmatrix}}_{A} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b. Use the general solution to find a basis for the subspace of solutions to $Ax = 0$.

A general solution is a description of all solutions using parameters.

Exercise 2

a. Find the general solution to the following matrix equation.

$$\underbrace{\begin{bmatrix} 2 & -2 & -4 & 4 \\ -1 & 1 & 3 & 2 \end{bmatrix}}_{A} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

I set the 2nd and 4th var x and z to parameters

(a) Find all solutions by row reducing

$$\left[\begin{array}{cccc|c} 2 & -2 & -4 & 4 & 0 \\ -1 & 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto \frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & -1 & -2 & 2 & 0 \\ -1 & 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto R_1 + R_2} \underbrace{\left[\begin{array}{cccc|c} 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right]}_{REF}$$

No leading 1 on the 2nd and 4th col:

$\boxed{\begin{array}{l} \text{Let } x = t \\ \text{Let } z = s \end{array}}$

Back substitution:

$1.w - 1.x - 2.y + 2.z = 0 \Rightarrow w - t - 2(-4s) + 2s = 0 \Rightarrow w - t + 8s + 2s = 0$

$w - t + 10s = 0$

$1.y + 4z = 0 \Rightarrow y + 4s = 0 \Rightarrow \boxed{y = -4s}$ $\quad \boxed{w = t - 10s}$

So the general solution is $\begin{bmatrix} t - 10s \\ t \\ -4s \\ s \end{bmatrix}$ for t, s in $\mathbb{R}$.

b. Use the general solution to find a basis for the subspace of solutions to Ax = 0.

Comments • We just showed that every element in ker(A) is of the form

$$\begin{bmatrix} t - 10s \\ t \\ -4s \\ s \end{bmatrix} \underset{\uparrow}{=} \begin{bmatrix} t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -10s \\ 0 \\ -4s \\ s \end{bmatrix} = \underbrace{t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -10 \\ 0 \\ -4 \\ 1 \end{bmatrix}}$$

"separate t and s"

A linear combination of concrete vectors with t and s as coefficients

- So every element in ker(A) is a linear combination of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$.
- By def of spanning set, the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$ is a spanning set for ker(A).
- To be a basis of ker(A), a set needs to be both
  - a spanning set for ker(A)
  - a linearly independent set
- Check whether $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$ is linearly independent:

Row reduce

$$\left[\begin{array}{cc|c}1&-10&0\\1&0&0\\0&4&0\\0&1&0\end{array}\right] \xrightarrow[R_2 \mapsto -R_1+R_2]{} \left[\begin{array}{cc|c}1&-10&0\\0&10&0\\0&4&0\\0&1&0\end{array}\right] \xrightarrow[\substack{R_2 \mapsto \frac{1}{10}R_2 \\ R_3 \mapsto \frac{1}{4}R_3}]{} \left[\begin{array}{cc|c}1&-10&0\\0&1&0\\0&1&0\\0&1&0\end{array}\right] \xrightarrow[\substack{R_3 \mapsto -R_2+R_3 \\ R_4 \mapsto -R_2+R_4}]{} \underbrace{\left[\begin{array}{cc|c}1&-10&0\\0&1&0\\0&0&0\\0&0&0\end{array}\right]}_{\text{REF}}$$

Every column (left of "|") has a leading 1, so the vectors are linearly independent :)

— end of explanation —

So $\boxed{\left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-10\\0\\4\\1\end{bmatrix} \right\} \text{ is a basis for } \ker(A)}$ ← Short answer to Ex 2(b)

Extra info:

This means $\ker(A)$ has dimension 2, the same dimension as a plane in 3D.

So think of this space as a "plane" in 4D.

Algorithm 1 (basis for kernel): Find one basis for the kernel of A

1. Put A into REF.
2. Write a general solution to $Ax = 0$, introducing a parameter for each column of the REF without a leading 1.
3. Rewrite the general solution as a linear combination whose coefficients are the parameters.

Then vectors in the linear combination form a basis for $\ker(A)$.

Why does this work?

It's a spanning set since every solution is a linear combination.
It's linearly independent since each vector is non-zero in a new row

This is not the only basis for the kernel of A!

Exercise 3

$$A := \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

a. Find a basis for $\ker(A)$ and the dimension of $\ker(A)$.

b. Check that the following set of vectors is a basis for $\ker(A)$.

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Pause video & try on your own
(Use Algorithm 1 and follow Exercise 2)

Exercise 3

$$A := \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

a. Find a basis for $\ker(A)$ and the dimension of $\ker(A)$.

[Following Algorithm 1]

Step 1: Row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto -R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & -3 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R_3 \mapsto R_2 + R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \mapsto -R_2} \underbrace{\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{REF}}$$

Step 2: Find general solution

(Choose your own variable names $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ or $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ or $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$)

Let $\boxed{x_2 = t}$, $\boxed{x_4 = s}$.

Back substitution:

$x_1 + x_2 + x_3 + 2x_4 = 0 \Rightarrow x_1 + t + (-3s) + 2s = 0 \Rightarrow x_1 + t - s = 0$

$\boxed{x_1 = -t + s}$

$x_3 + 3x_4 = 0 \Rightarrow x_3 + 3s = 0 \Rightarrow \boxed{x_3 = -3s}$

General solution is $\begin{bmatrix} -t-s \\ t \\ -3s \\ s \end{bmatrix}$

Step 3: Write general solution as linear combination with parameters as coefficients

$$\begin{bmatrix} -t-s \\ t \\ -3s \\ s \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ -3s \\ s \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

$\therefore$ $\boxed{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \ker(A).}$

$\therefore$ $\boxed{\dim(\ker(A)) = 2.}$

~ the end of Ex 3 (a) ~

Exercise 3

$$A := \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

b. Check that the following set of vectors is a basis for $\ker(A)$.

$$L := \left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Since we already found $\dim(\ker(A)) = 2$ from part (a), let's apply the "2 out of 3 rule".

First, we have to check that L is a subset of $\ker(A)$:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark \quad \text{so } \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix} \text{ is in } \ker(A)$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark \quad \text{so } \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \text{ is in } \ker(A)$$

So L is contained in $\ker(A)$.

Next, since we've seen that $\dim(\ker(A)) = 2$ and L has two vectors, we just need to check that L spans $\ker(A)$

or

L is linearly independent.

(I'll check this)

$$L := \left\{ \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Row reduce

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -3 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_4} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -3 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto 3R_1 + R_3]{R_2 \mapsto -R_1 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_4 \mapsto R_2 + R_4} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto -R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

I got an REF matrix with a leading 1 in each column left of "|".

So L is linearly independent.

Combined with the fact that L is a subset of $\ker(A)$, L has $\dim(\ker(A))$-many vectors, this shows that

L is a basis for $\ker(A)$.

~ end of Ex 3(b) ~

If we can find a basis, we can find the dimension.

$$\begin{aligned}\text{dimension of } \ker(A) &\overset{\text{def}}{=} \#\text{ of vectors in basis}\\ &\overset{\text{Algo 1}}{=} \#\text{ of parameters in general solution}\\ &= \#\text{ columns in REF without a leading 1}\end{aligned}$$

That is...

Fact 2: The dimension of the kernel of A

$$\dim(\ker(A)) = \underbrace{\text{width}(A)}_{\#\text{ of columns of } A \text{ total}} - \underbrace{\text{rank}(A)}_{\#\text{ of columns of the REF with a leading 1}}$$

E.g. If A is 6×7 and $\text{rank}(A) = 2$, then $\dim(\ker(A)) = 7 - 2 = 5$

We have five "standard" types of subsets which we know are always subspaces.

We now have methods to find a basis and the dimension of each of our general constructions of a subspace!

Finding a basis and dimension for standard subspaces

	Subspace	Method to find one basis	Dimension
Algo 5 from Lec13b	1. Image of A	Columns with L1 in REF	rank
	2. Span of $\{v_1, \ldots v_n\}$	$=$ im(concatenation), use $\uparrow$	$\uparrow$
New, in this lecture	3. Kernel of A	Vectors in general solution	width $-$ rank
	4. Solutions to HSLE	$=$ ker(coeff. matrix), use $\uparrow$	$\uparrow$
	5. λ-eigenspace of A	$=$ ker(A $- \lambda$Id), use $\uparrow$	$\uparrow$

In each case, the dimension is easy if we know a certain rank.

For a general subspace you encounter in the wild which is not one of these five types, we usually don't have an easy algorithm for finding a basis.

Exercise 4

(a) Find a basis of the 2-eigenspace of

$$M := \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

(b) What is the dimension of this eigenspace?

(Recall) <u>Def</u> The λ-eigenspace of M is $\{v \text{ in } \mathbb{R}^{\text{width}(M)} \text{ where } Mv = \lambda v\}$

So the 2-eigenspace of M is $W := \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 \text{ where } \underbrace{M\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\left(M - 2\,\mathrm{Id}_{3\times 3}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \right\}$

Note $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ where } \begin{bmatrix} 2-2 & 2 & 4 \\ 0 & 1-2 & -2 \\ 0 & 1 & 4-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

i.e. $W = \ker\left(\begin{bmatrix} 0 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} \right).$

Algorithm 1 (Find a basis for the kernel of a matrix) says we just need to solve for $\begin{bmatrix} 0 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

write the solutions as linear combinations of a set S of vectors — the set S will be a basis for W.

Row reduce:

$$\left[\begin{array}{ccc|c} 0 & 2 & 4 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto -R_1 + R_3]{R_2 \mapsto R_1 + R_2} \underbrace{\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{REF}}$$

1st and 3rd columns have no leading 1

Let $\boxed{x = t}$

Let $\boxed{z = r}$

Note: If there had been k number of columns without a leading 1, $\dim(W) = k$

Back substitution:

$y + 2z = 0 \implies y + 2r = 0 \implies y = \boxed{-2r}$

General solution: $\begin{bmatrix} t \\ -2r \\ r \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2r \\ r \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.

a) $\therefore$ $\boxed{\text{A basis for } W \text{ is } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}}$.

b) So $\dim(W) = 2$.

Let's do a sanity check. Check that at least $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a subset of the 2-eigenspace of M.

Check: $M\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \stackrel{?}{=} 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $M\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \checkmark$$

So at least our set of two vectors is a subset of W.