Lecture 13b

Bases (Basis computations and dimensions)

Review

Definition Basis (plural: bases)

A **basis for a subspace V** is a linearly independent spanning set of V.

A subspace can have many different bases, but ...

Theorem 4 (The Invariance Theorem)

Any two bases for a subspace contain the same number of vectors.

Definition Dimension

The dimension of a subspace V is the number of vectors in any basis of V.

Goal for Lecture 13b

How to quickly find a basis for various types of subspaces (and compute the dimension of a subspace).

Fact (Bases from spanning sets)

Every spanning set for a subspace V contains a basis for V.

That is, you can make a basis by throwing out enough elements.

Example

$$\text{A spanning set of } \mathbb{R}^2\text{:} \quad \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 5 \\ 4 \end{bmatrix} \right\}$$

$$\text{One subset that is a basis:} \quad \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

$$\text{Another subset that is a basis:} \quad \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

Spanning sets and dimension

A spanning set for V must contain at least $\dim(V)$-many vectors.

The previous idea is pretty slow; we can speed it up.

Algorithm 5: Finding a basis from a spanning set

Let $\{v_1, v_2, \ldots, v_k\}$ be a spanning set for V.

- Concatenate the vectors into a matrix A.
- Put A into REF, call it B.
- Check which columns of B contain a leading 1
- Keep the vectors in the original set corresponding to <u>only</u> those columns.

The result is a basis for V.

Exercise 6 (Apply Algorithm 5)

Let

$$V := \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$$

Find a basis for V.

Exercise 6 (Apply Algorithm 5)

Let
$$V \overset{def}{:=} \operatorname{span}\left\{ \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix} \right\}$$
Find a basis for V.

By def, the set
$$S := \left\{ \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix} \right\}$$
is a spanning set for V.

(I will follow Algorithm 5)

Let
$$A := \underbrace{\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}}_{\text{concatenate the vectors } \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}}$$

(We row reduce the concatenation)

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} \xrightarrow[\text{an REF matrix}]{\text{Perform elementary row operations until you get}} \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

We get an REF matrix whose 1st and 2nd columns have leading 1s, not the 3rd column.

$$\boxed{\text{The set } \left\{ \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix} \right\} \text{ is a basis for } V.}$$

Reminder: This means the dimension of V is 2.

Recall fact: Span equals image of concatenation

For example,

$$\operatorname{span}\left\{\begin{bmatrix}1\\4\\7\\-1\end{bmatrix},\begin{bmatrix}2\\5\\8\\0\end{bmatrix},\begin{bmatrix}3\\6\\9\\2\end{bmatrix}\right\} = \operatorname{im}\left(\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\\-1 & 0 & 2\end{bmatrix}\right)$$

We can use this to reformulate the last argument.

Algorithm 5, rephrased: Finding a basis for an image

Let A be a matrix. One basis for $\operatorname{im}(A)$ is given by the columns of A in which an REF of A contains a leading 1.

This is just one of many bases for $\operatorname{im}(A)$!

Algorithm 5, rephrased: Finding a basis for an image

Let A be a matrix. One basis for $\mathrm{im}(A)$ is given by the columns of A in which an REF of A contains a leading 1.

Exercise 7(a) (Apply Algorithm 5, rephrased)

Let

$$M := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix}$$

Find a basis for $\mathrm{im}(M)$, the image of M.

Step 1: Row reduce M until we get an REF matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{REF}}$$

$R_2 \mapsto -R_1 + R_2$

$R_3 \mapsto -R_1 + R_3$

$R_4 \mapsto -R_1 + R_4$

$R_3 \mapsto -2R_2 + R_3$

$R_4 \mapsto -3R_2 + R_4$

Step 2: Note that only columns 1 and 2 of an REF of M have a leading 1.

By "Algorithm 5, rephrased",

$\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\} \text{ is a basis for } \mathrm{im}(M).}$

- Algorithm 5 tells us how to produce one basis for im(A): The number of vectors in this basis is the number of leading 1s in an REF of A, that is, the rank of A.
- If we know the number of vectors in one basis for im(A), we know the number of vectors in every basis, that is, the dimension. So ...

Theorem 6: Dimension of the image of a matrix

The dimension of the image of A is equal to the rank of A.

This is often used as the definition of rank(A).

Theorem 6: Dimension of the image of a matrix

The dimension of the image of A is equal to the rank of A.

Exercise 7(b) (Apply Theorem 6)

Find the dimension of the image of the following matrix. (Pretend you haven't seen the previous slide.)

$$M := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix}$$

(We need to compute the rank of M.)
Compute rank by first finding an REF matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix} \xrightarrow[\substack{R_2 \mapsto -R_1+R_2 \\ R_3 \mapsto -R_1+R_3 \\ R_4 \mapsto -R_1+R_4}]{} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \xrightarrow[\substack{R_3 \mapsto -2R_2+R_3 \\ R_4 \mapsto -3R_2+R_4}]{} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{REF}}$$

(An REF matrix equivalent to M has two leading 1s)

rank(M) = 2. So the dimension of im(M) is $\boxed{2}$.

of a subspace V of V

In Algorithm 5, we start with a spanning set S. Then construct a basis contained in S (by "keeping only the columns with a leading 1 in the REF").

Fact (Bases from linearly independent set)

Every linearly independent subset of V is contained in a basis for V.

Idea: We can start with a linearly independent set. If it's not already a basis, construct a basis by adding vectors which would turn it into a basis.

Example

$$\text{A linearly independent set, } S\text{: } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \text{ a subset of } \mathbb{R}^3$$

$$\text{A basis for } \mathbb{R}^3 \text{ containing } S\text{: } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Another basis for } \mathbb{R}^3 \text{ containing } S\text{: } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix} \right\}$$

Unlike Algorithm 5, we have no general algorithm for this. But it is still a useful fact.

Since every linearly independent subset of V is contained in some basis of V, we can say ...

Fact: Linearly independent sets and dimension

A linearly independent set in V has **at most** $\dim(V)$-many vectors.

Example

The following set cannot be linearly independent, as $\dim(\mathbb{R}^3) = 3$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} \right\}$$

Fact: A bound on dimension of subspaces

A subspace of $\mathbb{R}^n$ must have dimension at most n.

Reason: A basis for a subspace of $\mathbb{R}^n$ is linearly independent in $\mathbb{R}^n$.

Summary: Bounds on special sets

$$\#(\text{vectors in a } \underbrace{\text{LI}}_{\text{linearly independent}} \text{ set in } V) \leq \dim(V) \leq \#(\text{vectors in a } \underbrace{\text{SS}}_{\text{spanning set}} \text{ for } V)$$

We can say something stronger if there is equality.

Theorem 7 (Special sets of just the right size)

- If a spanning set for V has $\dim(V)$-many vectors (the smallest size possible for a spanning set), then it must be a basis.
- If a linearly independent subset of V has $\dim(V)$-many vectors (the largest size possible for a linearly independent subset), then it must be a basis.

Why?

Because you can't delete/add vectors and still have $\dim(V)$-many.

I find it useful to restate this in the following way.

Theorem 7, restated (The '2 out of 3 Rule' for checking a basis)

Let $v_1, v_2, \ldots, v_k$ be elements in V. If any 2 of the following 3 properties are true, then the 3rd one is automatically true.

- $v_1, v_2, \ldots, v_k$ is a spanning set.
- $v_1, v_2, \ldots, v_k$ is linearly independent.
- The dimension of V is k.

So, if any 2 of these are true, then $v_1, v_2, \ldots, v_k$ is a basis for V.

If you know $\dim(V)$ and you want to check if a subset S of V is a basis...

- If S has the wrong number of vectors, it's not basis.
- If S has the right number of vectors, you only need to check one of the two conditions (and one is usually easier).

Let's practice applying the '2 out of 3 rule'.

Exercise 8

Let W be the subspace of $\mathbb{R}^3$ consisting of vectors whose entries sum to 0. Let

$$T := \left\{ \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$$

We see that the vectors of T are in W. Use the '2 out of 3 rule' to check whether the set T is a basis for W.

Exercise 8 solution + Instructor's comments

We check: Both $\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ are in W $\left(\begin{matrix} -1+3-2=0 \\ 2-1-1=0 \end{matrix} \right)$. ✓

Recall: In Exercise 1 of Lecture 13a we showed that a set of two vectors is a basis for W. This means $\dim(W) = 2$.

Our set $T = \left\{ \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$ has two vectors (same number as $\dim(W)$).

By the "2 out of 3 rule", we only need to check

- Our set is linearly independent

  OR

- Our set is a spanning set for W

We only need to check one of them - not both!

Which one is easier to check? In this case, linear independence is slightly easier to check

We'll check that our set T is linearly independent:

(We need to show that the equation $x \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, i.e. $\underbrace{\begin{bmatrix} -1 & 2 \\ 3 & -1 \\ -2 & -1 \end{bmatrix}}_{\text{Concatenation of the vectors in our set}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

has one unique solution, the trivial solution $\begin{matrix} x = 0 \\ y = 0 \end{matrix}$)

Row reduce the augmented matrix

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -1 & 0 \\ -2 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto -R_1} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 3 & -1 & 0 \\ -2 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto 2R_1 + R_3]{R_2 \mapsto -3R_1 + R_2} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 5 & 0 \\ 0 & -5 & 0 \end{array}\right] \xrightarrow{R_3 \mapsto R_2 + R_3} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \mapsto \frac{1}{5} R_2} \underbrace{\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{\text{REF}}$$

An REF matrix equivalent to $x \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a leading one in each column to the left of the vertical line.

(So $x \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has one unique solution, the trivial solution $\begin{matrix} x = 0 \\ y = 0 \end{matrix}$)

This shows that our set $T = \left\{ \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$ is linearly independent.

By the "2 out of 3 rule", since T has $\overbrace{\dim(W)}^{2}$-many vectors, our set T is a basis of W.

(We can deduce that our set is a basis of W without needing to check that it is a spanning set for W!)

— the end of solution + instructor's comments —

Exercise 8

Let W be the subspace of $\mathbb{R}^3$ consisting of vectors whose entries sum to 0. Let

$$T := \left\{ \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\}$$

We see that the vectors of T are in W. Use the '2 out of 3 rule' to check whether the set T is a basis for W.

SAMPLE STUDENT HOMEWORK ANSWER

We check: Both $\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$ are in W $\left(\begin{matrix} -1+3-2=0 \\ 2-1-1=0 \end{matrix} \right)$. ✓

In Exercise 1 of Lecture 13a we showed that a set of two vectors is a basis for W. This means $\boxed{\dim(W) = 2}$.

We'll check that our set T is linearly independent:

Row reduce the augmented matrix

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -1 & 0 \\ -2 & -1 & 0 \end{array}\right] \xrightarrow[R_1 \mapsto -R_1]{} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 3 & -1 & 0 \\ -2 & -1 & 0 \end{array}\right] \xrightarrow[\substack{R_2 \mapsto -3R_1+R_2 \\ R_3 \mapsto 2R_1+R_3}]{} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 5 & 0 \\ 0 & -5 & 0 \end{array}\right] \xrightarrow[R_3 \mapsto R_2+R_3]{} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto \frac{1}{5}R_2]{} \underbrace{\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{\text{REF}}$$

An REF matrix equivalent to $x\begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a leading one in <u>each</u> column to the left of the vertical line.

So, T is linearly independent.

By the "2 out of 3 rule", since T has $\overbrace{\dim(W)}^{2}$-many vectors, our set T is a basis of W.

— end of Sample answer —

From class lecture on Friday, Nov 6

Exercise 9 Suppose a subspace V has a spanning set

$$S := \left\{ \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Find a basis for V (which is contained in S)

Solution

Algorithm 5: Finding a basis from a spanning set

Let $\{v_1, v_2, \ldots, v_k\}$ be a spanning set for V.

- Concatenate the vectors into a matrix A.
- Put A into REF, call it B.
- Check which columns of B contain a leading 1
- Keep the vectors in the original set corresponding to <u>only</u> those columns.

The result is a basis for V.

Let $$A := \underbrace{\begin{bmatrix} 6 & 1 & 6 & 0 & 0 & 2 \\ 7 & 2 & 7 & 1 & 1 & 0 \\ 8 & 3 & 8 & 2 & 1 & 1 \\ 9 & 4 & 0 & 3 & 1 & 1 \end{bmatrix}}_{\text{A concatenation of the vectors in } S}$$

$$\begin{bmatrix} 6 & 1 & 6 & 0 & 0 & 2 \\ 7 & 2 & 7 & 1 & 1 & 0 \\ 8 & 3 & 8 & 2 & 1 & 1 \\ 9 & 4 & 0 & 3 & 1 & 1 \end{bmatrix} \xrightarrow[\text{using a computer}]{\text{Row reduce}} \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{5} & 0 & \frac{4}{45} \\ 0 & 1 & 0 & \frac{1}{5} & 0 & 4/5 \\ 0 & 0 & 1 & 0 & 0 & 1/9 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

Call this B

Column 4 and column 6 of B have no leading 1.
So we remove the 4th and 6th column vectors of A from S

$$\left\{ \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \cancel{\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}} \right\}$$

So $\left\{ \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for V.

— the end —

Exercise 10

Suppose W is a subspace and we know $\dim(W) = 4$ (Note: this means any basis for W has $\underline{4}$ vectors).

Let $T := \left\{ \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be a subset of W.

Check whether T is a basis for W.

Solution

Theorem 7, restated (The '2 out of 3 Rule' for checking a basis)

Let $v_1, v_2, \ldots, v_k$ be elements in V. If any 2 of the following 3 properties are true, then the 3rd one is automatically true.

- $v_1, v_2, \ldots, v_k$ is a spanning set.
- $v_1, v_2, \ldots, v_k$ is linearly independent.
- The dimension of V is k.

So, if any 2 of these are true, then $v_1, v_2, \ldots, v_k$ is a basis for V.

- Check that the number of vectors in T is $\dim(W)$. ✓
- Check that T is a linearly independent set.

That is, check that

$$x \begin{bmatrix} 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + z \begin{bmatrix} 6 \\ 7 \\ 8 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

has one unique solution, the trivial solution $x=0$, $y=0$, $z=0$, $w=0$.

$$\begin{bmatrix} 6 & 1 & 6 & 0 \\ 7 & 2 & 7 & 1 \\ 8 & 3 & 8 & 1 \\ 9 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 6 & 1 & 6 & 0 & 0 \\ 7 & 2 & 7 & 1 & 0 \\ 8 & 3 & 8 & 1 & 0 \\ 9 & 4 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{cccc|c} 1 & * & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The REF having a leading 1 in every column to the left of the vertical line means the system has one unique solution.

Hence T is linearly independent.

$\boxed{\text{Since } \dim(W) \text{ is equal to the number of vectors in } T \text{ and } T \text{ is linearly independent, the set } T \text{ is a basis for } W.}$

— the end —