Lecture 13a

Bases

Review

Definition: Basis (plural: bases)

A **basis for a subspace** V is a linearly independent spanning set of V.

A basis is used to efficiently construct every element in a subspace.

Goldilocks and the three properties

A set of vectors $\{\mathrm{v}_1, \mathrm{v}_2, ..., \mathrm{v}_r\}$ in a subspace V is...

- ...a spanning set for V if every element of V can be written as a linear combination in at least one way (possibly more than one way),
- ...a linearly independent set if every element of V can be written as a linear combination in at most one way (possibly not every element of V is a linear combination of $v_1, v_2, \ldots, v_r$), and
- ...a basis for V if every element of V can be written as a linear combination in exactly one way.

Useful trick: Linear combination = matrix multiplication

$$\underbrace{c_1 v_1 + c_2 v_2 + \cdots + c_r v_r}_{\text{Linear combination of } v_1, v_2, \ldots, v_r} = \underbrace{\begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & \cdots & | \end{bmatrix}}_{\text{Concatenation}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{bmatrix}$$

Rule 1: Checking the three conditions

Let $v_1, v_2, \ldots, v_r$ be vectors in a subspace V, and let A be the concatenation of the vectors. Then the set $\{v_1, v_2, \ldots, v_r\}$ is...

1. ...a spanning set for V if, for each b in V, the equation $Ax = b$ is consistent.
2. ...linearly independent if, for all b in V, the equation $Ax = b$ has at most one solution; equivalently, A has rank equal to its width (the number of vectors, r).
3. ...a basis for V if, for all b in V, the equation $Ax = b$ has a unique solution.

Exercise 1

Let W be the subspace of $\mathbb{R}^3$ consisting of vectors whose entries sum to 0. Show that

$$S := \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$$

is a basis for W.

To answer, we apply Rule 1(3)

Let ① $\mathrm{v}_1, \mathrm{v}_2$ be vectors in a subspace W, and ② let A be the concatenation of the vectors. Then the set $\{\mathrm{v}_1, \mathrm{v}_2\}$ is...

- ...a basis for W if, ③ for all b in W, the equation $\mathrm{Ax} = \mathrm{b}$ has a unique solution.

① First, check that the vectors in S are in the subspace W.

Since $1 + (-1) + 0 = 0$, $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is in W.

Since $2 + 0 + (-2) = 0$, $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ is also in W.

Exercise 1

Let W be the subspace of $\mathbb{R}^3$ consisting of vectors whose entries sum to 0. Show that

$$S := \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$$

is a basis for W.

② Let $A := \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -2 \end{bmatrix}$.

③ Let b be in W. That is,

$b = \begin{bmatrix} a \\ c \\ d \end{bmatrix}$ for some a, c, d in $\mathbb{R}$ such that $a + c + d = 0$
(or $d = -a-c$)

In other words, $b = \begin{bmatrix} a \\ c \\ -a-c \end{bmatrix}$ for some a, c in $\mathbb{R}$.

To count solutions to $A\begin{bmatrix} x \\ y \end{bmatrix} = b$, we first row reduce the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ -1 & 0 & c \\ 0 & -2 & -a-c \end{array}\right] \xrightarrow[R_2 \mapsto R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 2 & a+c \\ 0 & -2 & -a-c \end{array}\right] \xrightarrow[R_3 \mapsto R_2 + R_3]{} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 2 & a+c \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto \frac{1}{2} R_2]{} \underbrace{\left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & \frac{a+c}{2} \\ 0 & 0 & 0 \end{array}\right]}_{\text{An REF matrix equivalent to } A\begin{bmatrix} x \\ y \end{bmatrix} = b}$$

The right column has no leading 1. This means $A\begin{bmatrix} x \\ y \end{bmatrix} = b$ is consistent (has at least one sol).

Each column to the left of the vertical line has leading 1.
This means $A\begin{bmatrix} x \\ y \end{bmatrix} = b$ has one unique solution.

So, for each b in W, the equation $A\begin{bmatrix} x \\ y \end{bmatrix}$ has a unique solution.

Therefore, S is a basis for W.

— the end —

Reminder

For any n, the set $\mathbb{R}^n$ is a subspace of itself.

Bases for $\mathbb{R}^n$ will be particularly interesting; let's do an example.

Exercise 2

Show that the following set is a basis for $\mathbb{R}^3$.

$$S := \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$$

To answer, we apply Rule 1(3)

Let $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3$ be vectors in a subspace W, and let A be the concatenation of the vectors. Then the set $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$ is...

- ...a basis for W if, for all b in W, the equation $\mathrm{Ax} = \mathrm{b}$ has a unique solution.

Here W is the entire $\mathbb{R}^3$.

Exercise 2

Show that the following set is a basis for $\mathbb{R}^3$.

$$S := \left\{ \begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}4\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix} \right\}$$

Let $A := \begin{bmatrix}2&4&2\\0&1&2\\1&2&2\end{bmatrix}$. $\Big\}$ Concatenation of vectors in S

Let b be in $\underbrace{\mathbb{R}^3}_{\text{our subspace } W}$,

that is, $b = \begin{bmatrix}a\\c\\d\end{bmatrix}$ for some a, c, d in $\mathbb{R}$.

We need to count solutions to $A\begin{bmatrix}x\\y\\z\end{bmatrix} = b$.

Row reduce the augmented matrix:

$$\left[\underbrace{\begin{matrix}2&4&2\\0&1&2\\1&2&2\end{matrix}}_{A}\;\middle|\;\underbrace{\begin{matrix}a\\c\\d\end{matrix}}_{b}\right] \xrightarrow[R_1 \mapsto \frac{1}{2}R_1]{} \left[\begin{array}{ccc|c}1&2&1&\frac{a}{2}\\0&1&2&c\\1&2&2&d\end{array}\right] \xrightarrow[R_3 \mapsto -R_1+R_3]{} \left[\begin{array}{ccc|c}\text{\textcircled{1}}&2&1&\frac{a}{2}\\0&\text{\textcircled{1}}&2&c\\0&0&\text{\textcircled{1}}&d-\frac{a}{2}\end{array}\right]$$

$\Big\}$ An REF matrix equivalent to $A\begin{bmatrix}x\\y\\z\end{bmatrix} = b$

We have a leading 1 in every column to the left of the vertical line and no leading 1 in the right-most column.

So $A\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}a\\c\\d\end{bmatrix}$ has one unique solution.

Therefore, the vectors in S form a basis for $\mathbb{R}^3$.

— the end —

Rule 1 for $\mathbb{R}^n$: Checking the three conditions using rank

Let $v_1, v_2, \ldots, v_m$ be vectors in $\mathbb{R}^n$, and let A be the concatenation of the vectors. Then the set $\{v_1, v_2, \ldots, v_m\}$ is...

- ...a spanning set for $\mathbb{R}^n$ if $\operatorname{rank}(A) = \operatorname{height}(A)$.
  Why? Because for $Ax = b$ to be consistent I would need the augmented matrix $[A|b]$ to have an REF with a leading 1 in every row (on the left of the vertical line). Otherwise I will be able to find a vecor b where the REF will have a leading 1 in the right column. e.g $\left[\begin{array}{ccc|c} 1 & \cdot & \cdot & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \triangle \end{array}\right] \leftarrow$ some non zero number
- ...linearly independent if $\operatorname{rank}(A) = \operatorname{width}(A)$.
  (From the last lecture)
- ... a basis for $\mathbb{R}^n$ if $\operatorname{rank}(A) = \operatorname{height}(A) = \operatorname{width}(A)$.

These conditions only work for bases of the subspace $\mathbb{R}^n$ (as a subspace of itself), not other subspaces of $\mathbb{R}^n$!

Rule 1 for $\mathbb{R}^n$: Rephrased in terms of rank

Let $v_1, v_2, \ldots, v_m$ be vectors in $\mathbb{R}^n$, and let A be the concatenation of the vectors. Then the set $S := \{v_1, v_2, \ldots, v_m\}$ is...

- ... a basis for $\mathbb{R}^n$ if $\mathrm{rank}(A) = \mathrm{height}(A) = \mathrm{width}(A)$.

The height of A is the height of the vectors in S. The vectors in S are in $\mathbb{R}^n$, so $\mathrm{height}(A) = n$.

Alternative solution to Exercise 2 (Using "Rule 1 for $\mathbb{R}^n$"):

- A concatenation A of the three vectors in S is a 3×3 matrix.
- Compute the determinant of A, get a nonzero number, and conclude A is invertible. Hence $\mathrm{rank}(A) = 3$.
- Since $\mathrm{rank}(A) = 3$ is equal to the width and height of A, "Rule 1 for $\mathbb{R}^n$" says that S is a basis for $\mathbb{R}^3$.

This condition only work for bases of the subspace $\mathbb{R}^n$ (as a subspace of itself), not other subspaces of $\mathbb{R}^n$!

"Rule 1 for $\mathbb{R}^n$" says ...

Theorem 2 (Rank and bases for $\mathbb{R}^n$)

A set of vectors in $\mathbb{R}^n$ is basis of $\mathbb{R}^n$ if its concatenation A has rank n.

We've seen: the rank of an $n \times n$ matrix is n if and only if it is invertible!

Theorem 3 (Invertibility and bases for $\mathbb{R}^n$)

The columns of an $n \times n$-matrix form a basis for $\mathbb{R}^n$ if and only if the matrix is invertible.

Alternative solution to Exercise 2 (using Thereom 3):

- A concatentation A of the three vectors in S is a 3×3 matrix.
- Compute the determinant of A, get a nonzero number, and conclude A is invertible. By Theorem 3, the vectors in S form a basis for $\mathbb{R}^3$.

These theorems only work for bases of the subspace $\mathbb{R}^n$ (as a subspace of itself), not other subspaces of $\mathbb{R}^n$!

Can we apply Theorem 2 or Theorem 3 to write an alternative solution to Exercise 1? NO

Exercise 3

Show that the standard basis vectors in $\mathbb{R}^3$ are a basis for $\mathbb{R}^3$.

The standard basis vectors in $\mathbb{R}^3$ are $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

We will apply Thm 3. First check if the concatenation $\begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}$ is invertible.

The concatenation is $Id_{3x3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We know Id_{3x3} is invertible (its inverse is itself).

Since the concatenation $\begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix} = Id_{3x3}$ is invertible,

Thm 3 tells us that $\{e_1, e_2, e_3\}$ form a basis for $\mathbb{R}^3$.

— the end —

The set of standard basis vectors is also called the standard basis for $\mathbb{R}^n$

The standard basis vectors in $\mathbb{R}^n$ always form a basis for $\mathbb{R}^n$.

Exercise 4

If the columns of A are a basis of $\mathbb{R}^n$, then the columns of $A^\top$ form a basis of $\mathbb{R}^n$.

Suppose the columns of A form a basis of $\mathbb{R}^n$.

Then A must be an $n \times n$ matrix by "Rule 1 for $\mathbb{R}^n$" (height(A) = width(A)).

Thm 3 says A must be invertible. So $\det(A) \neq 0$. (Note: det(A) is defined because A is square)

We know that $\det(A^T) = \det(A) \neq 0$. So A^T is invertible.

By Thm 3, the set of columns of A^T is a basis for $\mathbb{R}^n$.

Because $\mathrm{height}(A) = \mathrm{width}(A)$, we can also observe that...

The Invariance Theorem for $\mathbb{R}^n$

Every basis for $\mathbb{R}^n$ must have *n*-many vectors.

Example:

- If you are given a set of three vectors in $\mathbb{R}^4$, then you can immediately say that the set is not a basis for $\mathbb{R}^4$. (Too few)
- If you are given a set of five vectors in $\mathbb{R}^4$, then you can immediately say that the set is not a basis for $\mathbb{R}^4$. (Too many to be linearly independent)
- If you are given a set of four vectors in $\mathbb{R}^4$, then you need to do more computation to determine whether it is a basis for $\mathbb{R}^4$. (Correct # of vectors. Need to do computation)

Example: Several bases for $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 1 \\ 9 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 5 \end{bmatrix} \right\}$$

This is a special case of a deep property of bases.

Theorem 4 (The Invariance Theorem)

Any two bases for a subspace contain the same number of vectors.

This number is extremely useful, so we give it a name.

Definition: Dimension

The dimension of a subspace V is the number of vectors in any basis of V.

Examples

- $\dim(\mathbb{R}^3) = 3$.
- Let W be the subspace of 3-vectors whose entries sum to 0. Then $\dim(W) = 2$. (Exercise 1)
- Let V be the subspace of 3-vectors whose entries are the same. Then $\dim(V) = 1$. (Lecture 12b)

The algebraic definition of dimension is meant to generalize the notion of dimension in 3D or lower dimension.

Relation to geometry

This definition coincides with the geometric notion of dimension!

- The origin in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a subspace of dimension 0.
- A line through the origin is a subspace of dimension 1.
- A plane the origin is a subspace of dimension 2.

through

Exercise 5

For each set, determine whether it is a basis for $\mathbb{R}^3$.

$$S_1 := \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\0\\2\end{bmatrix}, \begin{bmatrix}4\\2\\3\end{bmatrix} \right\}, \quad S_2 := \left\{ \begin{bmatrix}1\\9\\5\end{bmatrix}, \begin{bmatrix}3\\6\\5\end{bmatrix}, \begin{bmatrix}7\\3\\5\end{bmatrix} \right\}$$

$$S_3 := \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\0\\2\end{bmatrix}, \begin{bmatrix}4\\2\\3\end{bmatrix}, \begin{bmatrix}5\\4\\1\end{bmatrix} \right\}, \quad S_4 := \left\{ \begin{bmatrix}1\\9\\5\end{bmatrix}, \begin{bmatrix}3\\6\\5\end{bmatrix} \right\}$$

(Pause the video and answer these before checking the solution.)

A concatenation of the vectors in S_1

is $A := \begin{bmatrix}1&3&4\\2&0&2\\1&2&3\end{bmatrix}$.

Compute $\det(A) = 2\cdot(-1)^{2+1}\det\begin{bmatrix}3&4\\2&3\end{bmatrix} + 2\cdot(-1)^{2+3}\cdot\det\begin{bmatrix}1&3\\1&2\end{bmatrix}$

$= -2\cdot(9-8) + -2\cdot(2-3)$

$= 0.$

Exercise 5

For each set, determine whether it is a basis for $\mathbb{R}^3$.

$$S_1 := \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\0\\2\end{bmatrix}, \begin{bmatrix}4\\2\\3\end{bmatrix} \right\}, \quad S_2 := \left\{ \begin{bmatrix}1\\9\\5\end{bmatrix}, \begin{bmatrix}3\\6\\5\end{bmatrix}, \begin{bmatrix}7\\3\\5\end{bmatrix} \right\}$$

$$S_3 := \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\0\\2\end{bmatrix}, \begin{bmatrix}4\\2\\3\end{bmatrix}, \begin{bmatrix}5\\4\\1\end{bmatrix} \right\}, \quad S_4 := \left\{ \begin{bmatrix}1\\9\\5\end{bmatrix}, \begin{bmatrix}3\\6\\5\end{bmatrix} \right\}$$

(Pause the video and answer these before checking the solution.)

Solution:

- The determinant of a concatenation of S_1 is 0, so it is not invertible. By Theorem 3, the set S_1 is not a basis for $\mathbb{R}^3$.
- The determinant of a concatenation of S_2 is nonzero, so it is invertible. By Theorem 3, the set S_2 **is** a basis for $\mathbb{R}^3$.
- The number of vectors in each of S_3 and S_4 is not 3, so they are not bases for $\mathbb{R}^3$.