Lecture 12b

Spanning Sets and Linear Independence

We now have several different ways to say the same thing.

Recall: Spanning sets

- $\{v_1, v_2, \ldots, v_n\}$ spans V.
- $\{v_1, v_2, \ldots, v_n\}$ is a spanning set for V.
- $V = \mathrm{span}\{v_1, v_2, \ldots, v_n\}$.
- $V = \mathrm{im}($the matrix whose column vectors are $v_1, v_2, \ldots, v_n)$.
- Every element of V can be written as a linear combination of the vectors $v_1, v_2, \ldots, v_n$ in at least one way.

Exercise 4 (review spanning sets)

Let W be the subset of vectors in $\mathbb{R}^3$ whose entries are the same. In Lecture 11b, Exercise 6, we showed that W is a subspace. Show that

$$\text{a}\quad \left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix} \right\} \text{ and } \quad \text{b} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

are spanning sets for W.

b Show $W = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$. Let v be in W. Then $v = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ for some a in $\mathbb{R}$.

We need to show that v is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, i.e.

$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = v$ has a solution.

$$\begin{bmatrix} c_1 \\ c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}.$$

$c_1 = a.$

Every v in W can be written as $v = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for some c_1 in $\mathbb{R}$.

Exercise 4 (review spanning sets)

Let W be the subset of vectors in $\mathbb{R}^3$ whose entries are the same. In Lecture 11b, Exercise 6, we showed that W is a subspace.

Show that

a $\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

are spanning sets for W.

a) Show $W = \text{span}\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix} \right\}$

Let v be in W. That is, $v = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ for some a in $\mathbb{R}$.

We need to show that v is a linear combination of $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}$,

i.e. there exist c_1, c_2, c_3 in $\mathbb{R}$ where

$$\underbrace{c_1 \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}}_{\text{combine into one vector}} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3c_1 + 7c_2 - 5c_3 \\ 3c_1 + 7c_2 - 5c_3 \\ 3c_1 + 7c_2 - 5c_3 \end{bmatrix}}_{\text{Write as a product of a matrix and a vector}} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$$

$$\begin{bmatrix} 3 & 7 & -5 \\ 3 & 7 & -5 \\ 3 & 7 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$$

Put into augmented matrix then row reduce

$$\left[\begin{array}{ccc|c} 3 & 7 & -5 & a \\ 3 & 7 & -5 & a \\ 3 & 7 & -5 & a \end{array}\right] \xrightarrow[R_3 \mapsto -R_1 + R_3]{R_2 \mapsto -R_1 + R_2} \left[\begin{array}{ccc|c} 3 & 7 & -5 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \mapsto \frac{1}{3}R_1} \underbrace{\left[\begin{array}{ccc|c} 1 & 7/3 & -5/3 & a/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{in row echelon form}}$$

We've shown $\begin{bmatrix} 3 & 7 & -5 \\ 3 & 7 & -5 \\ 3 & 7 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ is consistent (has at least one solution),

so v is a linear combination of $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}$, and $\begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}$.

— the end of a —

Spanning sets are good...

Finding a (finite) spanning set for a subspace allows us to easily construct every element of that subspace.

...but they could be better

If our spanning set is bigger than we need, this isn't an efficient construction.

Example (from Exercise 4)

Let W be the subset of vectors in $\mathbb{R}^3$ whose entries are the same. We just showed that both

$$\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

are spanning sets for W, but the former is less efficient. A single vector can be written as a linear combination in many ways:

$$1 \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + 0 \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix}$$

How can we measure how efficient a spanning set is?

How can we measure how efficient a spanning set is?

Idea

Measure efficiency by checking how many ways the zero vector can be written as a linear combination.

Def: The trivial linear combination of the set $\{v_1, v_2, \ldots, v_n\}$ is

$$0v_1 + 0v_2 + \cdots + 0v_n$$

DEFINITION 3: Linear independence

A set of vectors is **linearly independent** if the only linear combination which is equal to the zero vector is the trivial linear combination.

Def: A set of vectors is called linearly dependent if it is not linearly independent.

This definition makes sense for any set of vectors in $\mathbb{R}^n$.

Exercise 5

Show (a) that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is linearly independent and (b) that

$$\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ -5 \end{bmatrix} \right\}$$

is linearly dependent.

In fact, any set containing just one vector (as long as the vector is nonzero) is linearly independent.

Exercise 5

Show (a) that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is linearly independent and (b) that

Comments

We want to show if $a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

an arbitrary linear combination of a set of one vector

then $a = 0$

Answer

If $a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then $\begin{bmatrix} a \\ a \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

So $a = 0$.

So $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

— the end —

Exercise 5

Show (a) that

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$

is linearly independent and (b) that

$$\left\{\begin{bmatrix}3\\3\\3\end{bmatrix}, \begin{bmatrix}7\\7\\7\end{bmatrix}, \begin{bmatrix}-5\\-5\\-5\end{bmatrix}\right\}$$

is linearly dependent.

(b) We need to show that there are a, b, c in $\mathbb{R}$ (not all zeros) so that

$$\underbrace{a\begin{bmatrix}3\\3\\3\end{bmatrix} + b\begin{bmatrix}7\\7\\7\end{bmatrix} + c\begin{bmatrix}-5\\-5\\-5\end{bmatrix}}_{\text{combine into one vector}} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$\underbrace{\begin{bmatrix}3a+7b-5c\\3a+7b-5c\\3a+7b-5c\end{bmatrix}}_{\text{write as a product of a matrix and a vector}} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}3&7&-5\\3&7&-5\\3&7&-5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

This is a homogeneous SLE.

We know a homogeneous SLE has the zero vector as a solution (often called the trivial solution)

We want to know: Does $\begin{bmatrix}3&7&-5\\3&7&-5\\3&7&-5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ have more than just the trivial solution?

Turn into augmented matrix then row reduce

$$\left[\begin{array}{ccc|c}3&7&-5&0\\3&7&-5&0\\3&7&-5&0\end{array}\right] \xrightarrow[R_3 \mapsto -R_1+R_3]{R_2 \mapsto -R_1+R_2} \left[\begin{array}{ccc|c}3&7&-5&0\\0&0&0&0\\0&0&0&0\end{array}\right] \xrightarrow[R_1 \mapsto \frac{1}{3}R_1]{} \underbrace{\left[\begin{array}{ccc|c}1&7/3&-5/3&0\\0&0&0&0\\0&0&0&0\end{array}\right]}_{\text{Col 2 and col 3 have no leading 1s.}}$$

This shows that $\begin{bmatrix}3&7&-5\\3&7&-5\\3&7&-5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ has infinitely many solutions.

In particular, it has more than one solution,

so it has a non-zero solution.

Therefore, there is a nontrivial linear combination that is equal to the zero vector.

So the set $\left\{\begin{bmatrix}3\\3\\3\end{bmatrix}, \begin{bmatrix}7\\7\\7\end{bmatrix}, \begin{bmatrix}-5\\-5\\-5\end{bmatrix}\right\}$ is linearly dependent.

— the end —

Note: Along the way, we've shown that $\operatorname{rank}\left(\begin{bmatrix}3&7&-5\\3&7&-5\\3&7&-5\end{bmatrix}\right) = 1.$

We've stumbled on a useful way to check linear independence.

Concatenation

Given a set of vectors of the same height (in some order), the *concatenation* is the matrix with those column vectors.

E.g. The concatenation of $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\}$ is $\begin{bmatrix}1&4&1\\2&5&0\\3&6&2\end{bmatrix}$.

Theorem: Checking linear independence

A set of m vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbb{R}^n$ is linearly independent if the rank of their concatenation is m.

If the rank is less than m, the set is linearly dependent.

$\text{rank}\left(\begin{bmatrix}1&4&1\\2&5&0\\3&6&2\end{bmatrix}\right) = 3$, so $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\}$ is linearly independent.

the number of vectors

Exercise 6(a)

Determine whether the following set is linearly independent.

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$$

The concatenation is $M := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Compute rank(M) by row reduce.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[\text{until REF}]{\text{Row reduce}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

rank(M) = 2

Two is smaller than the number of vectors,

so $\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$ is linearly dependent.

Exercise 6(b)

Determine whether the following set is linearly independent.

$$\left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-3\\6\\1\end{bmatrix} \right\}$$

The concatenation is

$$M := \begin{bmatrix}1&1&1\\1&3&-3\\0&2&6\\0&1&1\end{bmatrix} \xrightarrow[R_2 \mapsto -R_1+R_2]{} \begin{bmatrix}1&1&1\\0&2&-4\\0&2&6\\0&1&1\end{bmatrix} \xrightarrow[\substack{R_2 \mapsto \frac{1}{2}R_2 \\ R_3 \mapsto \frac{1}{2}R_3}]{} \begin{bmatrix}1&1&1\\0&1&-2\\0&1&3\\0&1&1\end{bmatrix} \xrightarrow[\substack{R_3 \mapsto -R_2+R_3 \\ R_4 \mapsto -R_2+R_4}]{} \begin{bmatrix}1&1&1\\0&1&-2\\0&0&5\\0&0&3\end{bmatrix} \xrightarrow[\substack{R_3 \mapsto \frac{1}{5}R_3 \\ R_4 \mapsto \frac{1}{3}R_4}]{} \begin{bmatrix}1&1&1\\0&1&-2\\0&0&1\\0&0&1\end{bmatrix} \xrightarrow[R_4 \mapsto -R_3+R_4]{} \begin{bmatrix}1&1&1\\0&1&-2\\0&0&1\\0&0&0\end{bmatrix}$$

Three leading 1s

So $\operatorname{rank}(M) = 3$, which is equal to the number of vectors,

therefore $\left\{ \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\3\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-3\\6\\1\end{bmatrix} \right\}$ is linearly independent.

Exercise 6(c)

Why must the following set of vectors be linearly dependent?

$$\left\{ \begin{bmatrix}1\\5\\2\end{bmatrix}, \begin{bmatrix}3\\2\\8\end{bmatrix}, \begin{bmatrix}8\\5\\1\end{bmatrix}, \begin{bmatrix}1\\8\\3\end{bmatrix} \right\}$$

There are four vectors in this set.

But $\operatorname{rank}\left(\begin{bmatrix}1&3&8&1\\5&2&5&8\\2&8&1&3\end{bmatrix}\right) \leq 3$ because the number of leading 1s in an equivalent REF matrix is at most the height/width of the matrix. whichever is smaller

Since the rank of the concatenation of the set of vectors is smaller than the number of vectors in the set, the set must be linearly dependent.

Alternative answer: Compute the rank of the concatenation, which is 3. Since 3 is smaller than the number of vectors (4), the vectors must be linearly dependent.

Linear independence gives efficient linear combinations

Fact: If $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, then any vector can be written as a linear combination of $v_1, v_2, \ldots, v_n$ in at most one way.

either one way
or not possible

So, to efficiently construct vectors in a subspace, we need a...

DEFINITION 4: Basis

A **basis** for a subspace V is a spanning set of V which is linearly independent.

This is one of the most important definitions in the class. We will see bases have a lot of remarkable properties.

Example

The set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for the subspace of 3-vectors whose entries are the same.

How can I tell?

- Exercise 4 shows it is a spanning set of the vectors in $\mathbb{R}^3$ whose entries are the same
- Exercise 5 shows that it is linearly independent.

(possibly many ways, making this set too big)

We can restate these definitions more explicitly.

Goldilocks and the three properties

A set of vectors $\{v_1, v_2, \dots, v_n\}$ in a subspace V is...

- ...a spanning set for V if every element of V can be written as a linear combination in at least one way,
- ...a linearly independent set if every element of V can be written as a linear combination in at most one way, and
- ...a basis for V if every element of V can be written as a linear combination in exactly one way.

possibly no way (too small to be a spanning set)

Exercise 7

Show that

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$.

Comments

Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Two possible approaches:

- Check spanning set condition and linear independence separately.

But Exercise 4 & 5 show that the two steps require similar computation.

We'll use this approach

- Pet two cats with one hand:

Given an arbitrary vector in the subspace $\mathbb{R}^3$, check how many ways we can write v as a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

- If the answer is 0, then the vectors don't span the subspace.
- If the answer is more than one, then the vectors are not linearly independent.
- If the answer is exactly one, the vectors form a basis. :)

Answer to Exercise 7

Let v be in $\underbrace{\mathbb{R}^3}_{\text{the subspace in question}}$. That is, $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for some $\underbrace{a, b, c}_{\text{arbitrary fixed numbers}}$ in $\mathbb{R}$.

We wish to count the number of solutions to

$$\underbrace{x\begin{bmatrix}1\\0\\1\end{bmatrix} + y\begin{bmatrix}2\\1\\0\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix}}_{} = \underbrace{\begin{bmatrix}a\\b\\c\end{bmatrix}}_{\text{our } v}$$

Earlier I used c_1, c_2, c_3.
I switched to x, y, z
because they are easier to read.

write as the product of a matrix and a vector

$$\underbrace{\begin{bmatrix}1&2&3\\0&1&2\\1&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}}_{}$$

Turn into an augmented matrix, then row reduce

$$\left[\begin{array}{ccc|c}1&2&3&a\\0&1&2&b\\1&0&1&c\end{array}\right] \xrightarrow[R_3 \mapsto -R_1+R_3]{} \left[\begin{array}{ccc|c}1&2&3&a\\0&1&2&b\\0&-2&-2&c-a\end{array}\right] \xrightarrow[R_3 \mapsto 2R_2+R_3]{} \left[\begin{array}{ccc|c}1&2&3&a\\0&1&2&b\\0&0&2&2b+c-a\end{array}\right] \xrightarrow[R_3 \mapsto \frac{1}{2}R_3]{} \underbrace{\left[\begin{array}{ccc|c}1&2&3&a\\0&1&2&b\\0&0&1&\frac{2b+c-a}{2}\end{array}\right]}_{\text{REF}}$$

Every column to the left of the vertical line has a leading 1.

This means the original system, $\begin{bmatrix}1&2&3\\0&1&2\\1&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$, has one unique solution.

So, for all v in $\mathbb{R}^3$, there is exactly one way to write

$$v = x\begin{bmatrix}1\\0\\1\end{bmatrix} + y\begin{bmatrix}2\\1\\0\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix}.$$

Therefore, the set of vectors $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

— the end —