Lecture 12a

Spanning Sets

Last time, we generalized lines and planes through the origin.

Recall: A subspace of $\mathbb{R}^n$

A **subspace** of $\mathbb{R}^n$ is a non-empty subset V of $\mathbb{R}^n$ which is

- closed under addition; that is,

$$\text{for all } \mathrm{v}, \mathrm{w} \text{ in } V, \text{ the sum } \mathrm{v} + \mathrm{w} \text{ is in } V, \text{ and}$$

- closed under scalar multiplication; that is,

$$\text{for all } \mathrm{v} \text{ in } V \text{ and } c \text{ in } \mathbb{R}, \text{ the product } c\mathrm{v} \text{ is in } V.$$

Recall: Constructions of four types of subspaces

- The solution set to a homogeneous SLE
- The kernel of a matrix
- Eigenspaces of a matrix
- The image of a matrix

Recall: Checking if vectors are in these subspaces

You can check if a vector v is in...

- ...the solution set of a SLE by plugging in the entries (arithmetic)
- ...the kernel of A by checking if Av is 0 (arithmetic)
- ...the λ-eigenspace of A by checking if Av is λv (arithmetic)
- ...the image of A by checking if Ax = v is consistent (row reduction)

Goal

Reduce the information of a subspace (an <u>infinite</u> set of vectors) to a <u>finite</u> set of vectors, called a **spanning set**.

For a solution set, we already know how to do this, by using parameters. So we have a good answer for kernels and eigenspaces.

Exercise 1 (motivating example)

Find every element of the kernel of

$$A := \begin{bmatrix} 2 & -2 & -4 & 4 \\ -1 & 1 & 3 & 2 \end{bmatrix}$$

Exercise 1 (motivating example)

Find every element of the kernel of

$$A := \begin{bmatrix} 2 & -2 & -4 & 4 \\ -1 & 1 & 3 & 2 \end{bmatrix}$$

That is, find all solutions to $A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Row reduce the augmented matrix $[A \mid \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}]$:

$$\left[\begin{array}{cccc|c} 2 & -2 & -4 & 4 & 0 \\ -1 & 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow[R_1 \mapsto \frac{1}{2} R_1]{} \left[\begin{array}{cccc|c} 1 & -1 & -2 & 2 & 0 \\ -1 & 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto R_1 + R_2]{} \underbrace{\left[\begin{array}{cccc|c} 1 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right]}_{\text{REF}}$$

So $A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is equivalent to $\begin{cases} a - b - 2c + 2d = 0 \\ c + 4d = 0 \end{cases}$

Since col 2 and col 4 have no leading 1s, let $\boxed{b := t, \; d := r.}$

$a - b - 2c + 2d = 0 \Rightarrow a - t - 2(-4r) + 2r = 0 \Rightarrow a - t + 8r + 2r = 0 \Rightarrow \boxed{a = t - 10r}$

$c + 4d = 0 \Rightarrow c + 4r = 0 \Rightarrow \boxed{c = -4r}$

Every vector in $\ker(A)$ is of the form $\begin{bmatrix} t - 10r \\ t \\ -4r \\ r \end{bmatrix}$ for t, r in $\mathbb{R}$.

That is, $\boxed{\ker(A) = \left\{ \begin{bmatrix} t - 10r \\ t \\ -4r \\ r \end{bmatrix} \text{ for } t, r \text{ in } \mathbb{R} \right\}}$.

Important observation from Exercise 1

Our answer is equivalent to saying that every element of ker(A) can be written as a linear combination of two vectors:

$$\begin{bmatrix} t - 10r \\ t \\ -4r \\ r \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -10 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

We can observe a similar phenomenon for images.

Exercise 2

$$B := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Show that every element of $\mathrm{im}(B)$ is a linear combination of the **column vectors** of B.

Recall (Def) $\mathrm{im}(B) = \{ v \text{ in } \mathbb{R}^{\#\text{ rows of } B}$ such that $v = Bw$ for some w in $\mathbb{R}^{\#\text{ columns of } B} \}$

Exercise 2

$$B := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Show that every element of $\mathrm{im}(B)$ is a linear combination of the **column vectors** of B.

Solution + Instructor's Comments

[When you see the phrase "Show that every ◯..." or "Show that for all ◯...", start your argument with "Let [a letter] be ◯..."]

Let v be in im(B).

[Next, write what it means for [your chosen letter] to be ◯... In this case, write what it means for v to be in im(B)]

That is, $v = Bw$ for some w in $\mathbb{R}^3$.

[Our goal is to show that v is a linear combination of $\begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}$.
That is, show that v can be written as
$v = \text{some number} \begin{bmatrix}1\\4\\7\end{bmatrix} + \text{some number} \begin{bmatrix}2\\5\\8\end{bmatrix} + \text{some number} \begin{bmatrix}3\\6\\9\end{bmatrix}$]

So $v = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for some a, b, c in $\mathbb{R}$.

$$= \begin{bmatrix} 1a + 2b + 3c \\ 4a + 5b + 6c \\ 7a + 8b + 9c \end{bmatrix}$$

$$= \begin{bmatrix} 1a \\ 4a \\ 7a \end{bmatrix} + \begin{bmatrix} 2b \\ 5b \\ 8b \end{bmatrix} + \begin{bmatrix} 3c \\ 6c \\ 9c \end{bmatrix}$$

$v = a\begin{bmatrix}1\\4\\7\end{bmatrix} + b\begin{bmatrix}2\\5\\8\end{bmatrix} + c\begin{bmatrix}3\\6\\9\end{bmatrix}$ for some a, b, c in $\mathbb{R}$.

[The above sequence of equalities says
$v = a\begin{bmatrix}1\\4\\7\end{bmatrix} + b\begin{bmatrix}2\\5\\8\end{bmatrix} + c\begin{bmatrix}3\\6\\9\end{bmatrix}$ for some numbers a, b, c.
col vectors of B
We've shown v is a linear combination of the column vectors of B
We should tell the reader.]

Therefore, v is a linear combination of the column vectors of B.

[To conclude, write that we've shown the original statement]

We have shown that every element of im(B) is a linear combination of the column vectors of B. —the end—

Exercise 2

$$B := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Show that every element of $\mathrm{im}(B)$ is a linear combination of the **column vectors** of B.

SAMPLE STUDENT PROOF

Let v be in im(B).

That is, $v = Bw$ for some w in $\mathbb{R}^3$.

So $v = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for some a, b, c in $\mathbb{R}$.

$$= \begin{bmatrix} 1a + 2b + 3c \\ 4a + 5b + 6c \\ 7a + 8b + 9c \end{bmatrix}$$

$$= \begin{bmatrix} 1a \\ 4a \\ 7a \end{bmatrix} + \begin{bmatrix} 2b \\ 5b \\ 8b \end{bmatrix} + \begin{bmatrix} 3c \\ 6c \\ 9c \end{bmatrix}$$

$v = a \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + b \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + c \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ for some a, b, c in $\mathbb{R}$.

Therefore, v is a linear combination of the column vectors of B.

We have shown that every element of im(B) is a linear combination of the column vectors of B.

— the end of student's proof —

Exercise 1 says: we can write every element in the kernel of a matrix as a linear combination of a set of vectors.

Exercise 2 says: we can write every element in the image of a matrix as a linear combination of a set of vectors.

We want to do this for all subspaces.

DEFINITION 1: Span

The **span** of a set of vectors is the set of their linear combinations.

in this class, the set of vectors is usually finite

Example

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\} := \left\{ t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -10 \\ 0 \\ -4 \\ 1 \end{bmatrix} \text{ for all } t, s \text{ in } \mathbb{R} \right\}$$

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\} := \left\{ r \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + s \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + t \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \text{ for all } r, s, t \text{ in } \mathbb{R} \right\}$$

Fact 1: Spans are images of matrices

The image of A equals the span of the set of column vectors of A.

Example

(from Exercise 2)

$$\operatorname{im}\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$$

(of a matrix)

Since every image is a subspace, we get a result for free.

Fact 2: Spans are subspaces

The span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Fact 3: Subspaces are closed under spans

If a subspace contains a set of vectors, it also contains their span.

For example, if v_1, v_2, v_3 are in a subspace S, then we know that every linear combination of v_1, v_2, v_3 is also in S.

It is easy to construct every element of a span, and so we will often want to write a subspace as the span of a set of vectors.

DEFINITION 2: Spanning sets

A **spanning set** of V is a set of vectors whose span is V.

a subspace

'S spans V' $\Leftrightarrow$ S is a spanning set for V $\Leftrightarrow$ V$=$ span(S)

Exercise 3

Let W be the subspace of $\mathbb{R}^3$ consisting of vectors whose second entry is the average of the other two. Show that

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$$

is a spanning set for W.

Exercise 3

Let W be the subspace of $\mathbb{R}^3$ consisting of vectors whose second entry is the average of the other two. Show that

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is a spanning set for W.

Solution + Instructor's Comments

We need to show $W = \text{span}\left(\left\{ \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix} \right\} \right)$

Let v be in W. ← In Ex 2, this w was $\text{im}(B)$

[Next, write what it means for v to be in W]

That is, $v = \begin{bmatrix} a \\ \frac{a+b}{2} \\ b \end{bmatrix}$ for some numbers a, b

[We need to show that v is a linear combination of $\begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix}$.
That is, we need to show that there are c_1, c_2, c_3 such that
$$v = c_1 \begin{bmatrix}1\\4\\7\end{bmatrix} + c_2 \begin{bmatrix}2\\5\\8\end{bmatrix} + c_3 \begin{bmatrix}3\\6\\9\end{bmatrix}$$]

We want to show that the equation

$$c_1 \begin{bmatrix}1\\4\\7\end{bmatrix} + c_2 \begin{bmatrix}2\\5\\8\end{bmatrix} + c_3 \begin{bmatrix}3\\6\\9\end{bmatrix} = \begin{bmatrix} a \\ \frac{a+b}{2} \\ b \end{bmatrix}$$

has a solution.

[Warning: The variables are c_1, c_2, c_3.
The letters a, b represent fixed numbers.]

$$\begin{bmatrix} 1c_1 + 2c_2 + 3c_3 \\ 4c_1 + 5c_2 + 6c_3 \\ 7c_1 + 8c_2 + 9c_3 \end{bmatrix} = \begin{bmatrix} a \\ \frac{a+b}{2} \\ b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ \frac{a+b}{2} \\ b \end{bmatrix}$$

We will show that this equation has a solution, i.e. consistent.

Row reduce augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 4 & 5 & 6 & \frac{a+b}{2} \\ 7 & 8 & 9 & b \end{array}\right] \xrightarrow[R_3 \mapsto -7R_1+R_3]{R_2 \mapsto -4R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -3 & -6 & -4a + \frac{a+b}{2} \\ 0 & -6 & -12 & -7a+b \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -3 & -6 & \frac{-7a+b}{2} \\ 0 & -6 & -12 & -7a+b \end{array}\right] \xrightarrow[R_3 \mapsto -2R_2+R_3]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & -3 & -6 & \frac{-7a+b}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto \frac{-1}{3}R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 2 & \frac{-7a+b}{-6} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

[We are only interested in showing the system is consistent.
So we don't need to find the solutions
(although you can find the solutions using the usual back sub method)]

There is no leading 1 in the right column of an REF matrix equivalent to

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ \frac{a+b}{2} \\ b \end{bmatrix}.$$

This tells us that the system is consistent.

[That is, there exist c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} a \\ \frac{a+b}{2} \\ b \end{bmatrix}$$
]

So v can be written as a linear combination of $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$.

Therefore, the set $\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$ spans W.

[Meaning $W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right)$.]

— the end —

Exercise 3

Let W be the subspace of $\mathbb{R}^3$ consisting of vectors whose second entry is the average of the other two. Show that

$$\left\{ \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix} \right\}$$

is a spanning set for W.

SAMPLE STUDENT PROOF

We need to show $W = \text{span}\left(\left\{ \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix} \right\} \right)$

Let v be in W. That is, $v = \begin{bmatrix}a\\ \frac{a+b}{2}\\ b\end{bmatrix}$ for some numbers a, b.

We need to show that the equation

$$c_1 \begin{bmatrix}1\\4\\7\end{bmatrix} + c_2 \begin{bmatrix}2\\5\\8\end{bmatrix} + c_3 \begin{bmatrix}3\\6\\9\end{bmatrix} = \begin{bmatrix}a\\ \frac{a+b}{2}\\ b\end{bmatrix}$$ has a solution.

$$\begin{bmatrix}1c_1 + 2c_2 + 3c_3\\ 4c_1 + 5c_2 + 6c_3\\ 7c_1 + 8c_2 + 9c_3\end{bmatrix} = \begin{bmatrix}a\\ \frac{a+b}{2}\\ b\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}a\\ \frac{a+b}{2}\\ b\end{bmatrix}$$

We will show that this equation has a solution, i.e. consistent.

Row reduce augmented matrix:

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & a\\ 4 & 5 & 6 & \frac{a+b}{2}\\ 7 & 8 & 9 & b\end{array}\right] \xrightarrow[R_3 \mapsto -7R_1+R_3]{R_2 \mapsto -4R_1+R_2} \left[\begin{array}{ccc|c}1 & 2 & 3 & a\\ 0 & -3 & -6 & -4a + \frac{a+b}{2}\\ 0 & -6 & -12 & -7a+b\end{array}\right]$$

$$\left[\begin{array}{ccc|c}1 & 2 & 3 & a\\ 0 & -3 & -6 & \frac{-7a+b}{2}\\ 0 & -6 & -12 & -7a+b\end{array}\right] \xrightarrow{R_3 \mapsto -2R_2+R_3} \left[\begin{array}{ccc|c}1 & 2 & 3 & a\\ 0 & -3 & -6 & \frac{-7a+b}{2}\\ 0 & 0 & 0 & 0\end{array}\right] \xrightarrow{R_2 \mapsto \frac{-1}{3}R_2} \left[\begin{array}{ccc|c}1 & 2 & 3 & a\\ 0 & 1 & 2 & \frac{-7a+b}{-6}\\ 0 & 0 & 0 & 0\end{array}\right]$$

There is no leading 1 in the right column of an REF matrix equivalent to

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}a\\ \frac{a+b}{2}\\ b\end{bmatrix}.$$

This tells us that the system is consistent.

So v can be written as a linear combination of $\begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}$, and $\begin{bmatrix}3\\6\\9\end{bmatrix}$.

Therefore, the set $\left\{ \begin{bmatrix}1\\4\\7\end{bmatrix}, \begin{bmatrix}2\\5\\8\end{bmatrix}, \begin{bmatrix}3\\6\\9\end{bmatrix} \right\}$ spans W.

— the end —

We now have several different ways to say the same thing.

Equivalent statements

- $\{\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n\}$ spans V.
- $\{\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n\}$ is a spanning set for V.
- $V = \mathrm{span}\{\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n\}$.
- $V = \mathrm{im}($the matrix whose column vectors are $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n)$.
- Every element of V can be written as a linear combination of the vectors $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n$ in at least one way.

Here, V is a subspace