Lecture 11b

Subspaces, how to write proofs

Review

Definition: A subspace of $\mathbb{R}^n$

A **subspace** of $\mathbb{R}^n$ is a non-empty subset V of $\mathbb{R}^n$ which is

1. closed under addition; that is,

$$\text{for all } \mathsf{v}, \mathsf{w} \text{ in } V, \text{ the sum } \mathsf{v} + \mathsf{w} \text{ is in } V, \text{ and}$$

2. closed under scalar multiplication; that is,

$$\text{for all } \mathsf{v} \text{ in } V \text{ and } c \text{ in } \mathbb{R}, \text{ the product } c\mathsf{v} \text{ is in } V.$$

Last time: examples of subspaces

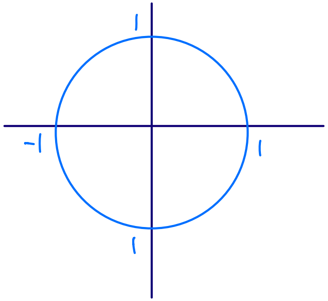
- The solution set to a homogeneous SLE.
- The **kernel** of a matrix
- An **eigenspace** of a square matrix
- The **image** of a matrix

Showing not a subspace

How do we show a subset S of $\mathbb{R}^n$ is not a subspace?
We just need to find one specific counterexample to one of the properties.

Exercise 4 (a)

Show that the unit circle in the plane is not a subspace of $\mathbb{R}^2$.



Exercise 4 (a)

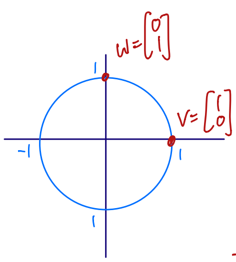
Show that the unit circle in the plane is not a subspace of $\mathbb{R}^2$.

SOLUTION + EXTRA COMMENTS FROM THE INSTRUCTOR

Let S be the set of points (vectors in $\mathbb{R}^2$) on the unit circle.

Strategy. To show that a subset of $\mathbb{R}^2$ is not a subspace, it's enough to show one of the following three things

- S is empty (cannot show this because S is not empty — there are points on the unit circle)
- S is not closed under addition (Let's try to show this)
- S is not closed under scalar multiplication



Let $v=\begin{bmatrix}1\\0\end{bmatrix}$ and $w=\begin{bmatrix}0\\1\end{bmatrix}$. They are both in S.

Then $v+w=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$, which is not in S.

Therefore, the unit circle is not closed under addition.

So the unit circle is not a subspace.

— the end of Solution + Extra Comments —

$\boxed{\text{Exercise 4(b)}}$ Show that the unit circle is not closed under scalar multiplication.

Exercise 4 (a)

Show that the unit circle in the plane is not a subspace of $\mathbb{R}^2$.

Sample Student Answer

Let S be the set of points (vectors in $\mathbb{R}^2$) on the unit circle.

We will show that S is not closed under addition.

Let $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. They are both in S.

Then $v + w = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which is not in S.

Therefore, the unit circle is not closed under addition.

So the unit circle is not a subspace.

— the end of sample student answer —

Showing a subspace

How do we show a subset S of $\mathbb{R}^n$ is a subspace?
We need to show that S is nonempty and that the two subset properties **always** hold.

We have to do this for every possible case, so we cannot simply check specific examples.

We must work with general cases (that is, abstractly).

A word on working abstractly

We work with arbitrary objects (like numbers, vectors, and sets) with specific properties, rather than concrete objects.

Instead of working with...	...we might work with...
the number 5	a solution x to an equation
the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	a vector v in a subspace
the set $\mathbb{R}^n$	a set V with a property

Why work abstractly?

- Often in math, we find an argument that works in many different situations.
- Instead of repeating the same argument again and again, we can give a general argument.

Such general arguments are called proofs.

Example 5

Let W be the subset of vectors in $\mathbb{R}^3$ whose entries are the same. Is W closed under addition? That is, is the sum of two elements of W always in W?

Examples suggest it might be true that the sum of two elements of W is always in W:

$$\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}3\\3\\3\end{bmatrix}=\begin{bmatrix}4\\4\\4\end{bmatrix}, \qquad \begin{bmatrix}-1\\-1\\-1\end{bmatrix}+\begin{bmatrix}2\\2\\2\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

But two examples are not enough! To show "the sum of two elements of W is in W" for **all** pairs in W, we need a general argument.

Example proposition (Example 5 con't)

Let W be the subset of $\mathbb{R}^3$ whose entries are the same. Then W is closed under addition.

Example proof

Let v and w be in W.

This means $v = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ and $w = \begin{bmatrix} b \\ b \\ b \end{bmatrix}$ for some a, b in $\mathbb{R}$.

(Next, we want to show that $v+w$ is in W)

Then $v+w = \begin{bmatrix} a \\ a \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \\ a+b \end{bmatrix}$.

Since all entries of $v+w$ are the same, we see that $v+w$ is in W.

Therefore, W is closed under addition.

Example proposition (Example 5 con't)

Let W be the subset of $\mathbb{R}^3$ whose entries are the same. Then W is closed under addition.

Example proof (Typed)

Let v and w be in W. This means we can write

$$\mathrm{v} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} \text{ and } \mathrm{w} = \begin{bmatrix} b \\ b \\ b \end{bmatrix} \quad \text{for some numbers } a \text{ and } b \text{ in } \mathbb{R}.$$

$$\text{Then} \quad \mathrm{v} + \mathrm{w} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \\ a+b \end{bmatrix}.$$

Since all the entries are the same, $\mathrm{v} + \mathrm{w}$ is in W.
Therefore, W is closed under addition.

A proof skeleton

Clarify is important.

Start the argument by communicating what you are assuming.

‘Let v and w be in W.’

This explains to the reader what kinds of objects we are considering and what their properties are.

Don’t be afraid to use words to explain your computation and remind the reader what you are showing.

‘Since all the entries are the same, $\mathrm{v} + \mathrm{w}$ is in W.’

This explains what the computation actually showed.

Conclude the argument by restating what we have shown.

‘Therefore, W is closed under addition.’

This connects the argument to the original problem.

Exercise 6

Let W be the subset of vectors in $\mathbb{R}^3$ whose entries are the same. Show that W is a subspace of $\mathbb{R}^3$. (We've done part of the proof already.)

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Proof with extra explanation from the Instructor

We will show that W is nonempty, closed under addition, and closed under scalar multiplication.

(First, show that W is nonempty. If you are not sure what vector to use, take the zero vector and show that it is in the subset in question)

Since $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in W, the set W is nonempty.

(Second, show that W is closed under addition)

Let v and w be in W. This means we can write

$$\mathrm{v} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} \text{ and } \mathrm{w} = \begin{bmatrix} b \\ b \\ b \end{bmatrix} \quad \text{for some numbers } a \text{ and } b \text{ in } \mathbb{R}.$$

$$\text{Then} \quad \mathrm{v} + \mathrm{w} = \begin{bmatrix} a \\ a \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \\ a+b \end{bmatrix}.$$

Since all the entries are the same, $\mathrm{v} + \mathrm{w}$ is in W.
Therefore, W is closed under addition.

(Finally, show that W is closed under scalar multiplication)

Start with the "let ..." sentence to communicate your assumption to the reader

Let v be in W and c in $\mathbb{R}$.

Write down what it means for v to be in W

This means $v = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ for some a in $\mathbb{R}$.

Our goal is to show that cv is in W, so we need to do some computation involving cv

↳ Then $cv = c\begin{bmatrix} a \\ a \\ a \end{bmatrix}$

$$= \begin{bmatrix} ca \\ ca \\ ca \end{bmatrix}.$$

This computation shows that all entries of cv are the same, so cv is in W. We need to tell this to the reader!

↳ Since every entry of cv is the same, the vector cv is in W.

Tell the reader what we have shown

↳ Therefore, W is closed under scalar multiplication.

We have shown :
- W is nonempty
- W is closed under addition
- W is closed under scalar multiplication.

So we are done with what we needed to show.

TELL THE READER WE ARE DONE

↳ We have shown that W is a subspace of $\mathbb{R}^3$.

— the end of proof with extra instructor's comments —

Exercise 6

Let W be the subset of vectors in $\mathbb{R}^3$ whose entries are the same. Show that W is a subspace of $\mathbb{R}^3$. (We've done part of the proof already.)

SAMPLE STUDENT PROOF

We will show that W is nonempty, closed under addition, and closed under scalar multiplication.

- Since $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in W, the set W is nonempty.
- Let v and w be in W.

  This means $v = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ and $w = \begin{bmatrix} b \\ b \\ b \end{bmatrix}$ for some a, b in $\mathbb{R}$.

  Then $v + w = \begin{bmatrix} a \\ a \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \\ a+b \end{bmatrix}$.

  Since all entries of $v+w$ are the same, we see that $v+w$ is in W.

  Therefore, W is closed under addition.
- Let v be in W and c in $\mathbb{R}$.

  This means $v = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$ for some a in $\mathbb{R}$.

  $$\text{Then } cv = c\begin{bmatrix} a \\ a \\ a \end{bmatrix} = \begin{bmatrix} ca \\ ca \\ ca \end{bmatrix}.$$

  Since every entry of cv is the same, the vector cv is in W.

Therefore, W is closed under scalar multiplication.

We have shown that W is a subspace of $\mathbb{R}^3$.

— the end of SAMPLE STUDENT PROOF —

Exercise 7 (Prove Theorem 2)

Prove that the kernel of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$ (directly from the definition of subspace).

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Prove that the kernel of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$ (directly from the definition of subspace).

Proof + Instructor's comments

Let A be an $m \times n$ matrix.

Recall def $\ker(A) \stackrel{\text{def}}{=} \left\{ v \text{ in } \mathbb{R}^n \text{ such that } Av = \underset{m \times 1}{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}} \right\}$

(First, show that $\ker(A)$ is nonempty. If you are not sure what vector to use, take the zero vector and show that it is in the subset in question)

Since $A \underset{n \times 1}{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}} = \underset{m \times 1}{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}$, the $n \times 1$ zero vector is in $\ker(A)$.

So $\ker(A)$ is nonempty.

(Next, show that $\ker(A)$ is closed under addition)

Let v and w be in $\ker(A)$. ← Communicate your assumption by writing "Let ..."

Then $Av = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ and $Aw = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ (*) ← Write down what it means for v and w to be in $\ker(A)$

Since we are now showing that $\ker(A)$ is closed under addition, we need to show that $v+w$ is in $\ker(A)$.
To show that $v+w$ is in $\ker(A)$, it's enough to show $A(v+w) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$. We need to now do some computation.

$$\begin{aligned} A(v+w) &= Av + Aw && \text{because matrix multiplication distributes over addition} \\ &= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} && \text{by } (*) \\ &= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \end{aligned}$$

We have done the computation we needed to show that $v+w$ is in $\ker(A)$. TELL THE READER.

Therefore, $v+w$ is in $\ker(A)$.

We've shown that $\ker(A)$ is closed under addition. Tell the reader.

So $\ker(A)$ is closed under addition.

(Finally, show $\ker(A)$ is closed under scalar multiplication)

Let v be in $\ker(A)$ and let c be in $\mathbb{R}$.

Then $Av \overset{(**)}{=} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

We should use this fact to show that cv is in $\ker(A)$.

To show that cv is in $\ker(A)$, we need to do some computation showing $A(cv) = \dots = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

We have $A(cv) = cAv$ scalar multiplication property

$= c\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ by $(**)$

$= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

This shows that cv is in $\ker(A)$.

So $\ker(A)$ is closed under scalar multiplication.

We have shown:
- $\ker(A)$ is nonempty
- $\ker(A)$ is closed under addition
- $\ker(A)$ is closed under scalar multiplication.

So we are done with what we needed to show.

TELL THE READER WE ARE DONE

We have shown that $\ker(A)$ is a subspace of $\mathbb{R}^n$.

— end of proof + Instructor's comments —

Exercise 7 (Prove Theorem 2)

Prove that the kernel of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$ (directly from the definition of subspace).

SAMPLE STUDENT PROOF

Let A be an $m \times n$ matrix.

Recall def $\ker(A) \overset{\text{def}}{=} \left\{ v \text{ in } \mathbb{R}^n \text{ such that } Av = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{m\times 1} \right\}$

- Since $A \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{m\times 1}$, the $n\times 1$ zero vector is in $\ker(A)$.

  So $\ker(A)$ is nonempty.

- Let v and w be in $\ker(A)$.

  Then $Av = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $Aw = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ $(*)$

  $A(v+w) = Av + Aw$ because matrix multiplication distributes over addition

  $= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ by $(*)$

  $= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Therefore, $v+w$ is in $\ker(A)$. So $\ker(A)$ is closed under addition.

- Let v be in $\ker(A)$ and let c be in $\mathbb{R}$.

  Then $Av \overset{(**)}{=} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

We have $A(cv) = cAv$ Scalar multiplication property

$= c \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ by $(**)$

$= \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

This shows that cv is in $\ker(A)$.

So $\ker(A)$ is closed under scalar multiplication.

We have shown that $\ker(A)$ is a subspace of $\mathbb{R}^n$.

— end of sample student proof —

(The same argument from above, typed)

A proof that $\ker(A)$ is a subspace

Since $A0 = 0$, the zero vector is in $\ker(A)$, i.e. $\ker(A)$ is non-empty.

Let v and w be in $\ker(A)$. Then $Av = 0$ and $Aw = 0$.
Since matrix multiplication distributes over addition, we have

$$A(v + w) = Av + Aw = 0 + 0 = 0$$

Thus, $v + w$ is in $\ker(A)$, and so $\ker(A)$ is closed under addition.

Next, let r be in $\mathbb{R}$, and let v be in $\ker(A)$. So $Av = 0$.
Then

$$A(rv) = r(Av) = r0 = 0$$

Therefore, rv is in $\ker(A)$, and so $\ker(A)$ is closed under scalar multiplication.

Hence, $\ker(A)$ is a subspace of $\mathbb{R}^n$.

Writing a proof is like building a bridge

It doesn't matter if it works once or twice. It needs to work in every possible case. Therefore, you must convince a skeptical reader that you have covered every possible case.

When in doubt, assume the reader is a confused classmate who has bet you $20 that you are wrong. Use standard language to avoid confusion. Clearly cover every case to leave no doubt.