Lecture 10b

Linear Transformations, part b

Recall: Theorem 1

Suppose $T : \underbrace{\mathbb{R}^n}_{\text{set of vectors of height } n} \to \underbrace{\mathbb{R}^m}_{\text{set of vectors of height } m}$. Then the following three statements are equivalent.

1. $T = T_A$ for some $m \times n$ matrix A; that is, T is a linear transformation.
2. T preserves addition and scalar multiplication.
3. T preserves linear combinations.

Goals:

- A trick to compute the matrix corresponding to a linear transformation.
- Geometric meaning of matrix algebra, determinant, and eigenvectors.

The standard basis vectors

Def: The *i*th **standard basis vector** in $\mathbb{R}^n$, denoted e_i, is the vector which is 1 in the ith entry and zero everywhere else.

Examples:

- The standard basis vectors in $\mathbb{R}^2$:

  $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- The standard basis vectors in $\mathbb{R}^3$:

  $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- The standard basis vectors in $\mathbb{R}^4$:

  $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Theorem: Finding the matrix of linear transformations

If a function $f : \mathbb{R}^n \to \overbrace{\mathbb{R}^m}^{\text{the set of vectors of height } m}$ is a linear transformation, then $f = T_A$, where

$$A := \overbrace{\begin{bmatrix} | & | & \cdots & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & \cdots & | \end{bmatrix}}^{A \text{ should have } n \text{ columns}} \left.\vphantom{\begin{bmatrix}|\\|\\|\end{bmatrix}}\right\} A \text{ should have } m \text{ rows}$$

That is, the columns of A are given by applying f to the standard basis vectors (of $\underbrace{\text{the appropriate size}}_{\text{in } \mathbb{R}^n}$).

Exercise 3

Suppose we know that the function $F: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and is given by the formula

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$$

Find a matrix A such that $F = T_A$. (We'll apply the above theorem)

Compute $F(e_i)$ for all standard basis vectors e_i.

$F(e_1) = F\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1+0 \\ 1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ the 1st col of A

$F(e_2) = F\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0+1 \\ 0-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ the 2nd col of A

Let $A := \underbrace{\begin{bmatrix} F(e_1) & F(e_2) \end{bmatrix}}_{\text{A has 2 columns}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

A has 2 columns because the domain of F is $\mathbb{R}^2$

By the theorem, $\boxed{F = T_A}$

Check that $T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$:

$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$

Exercise 4

Let L be the line through the points $(0,0)$ and (a,b), and let $F:\mathbb{R}^2 \to \mathbb{R}^2$ be projection onto L.

where $(a,b) \neq (0,0)$

1. Show that F is a linear transformation.
2. Find a matrix A such that $F = T_{\mathrm{A}}$.

① (Goal: We will show that F preserves addition (step 1) and scalar multiplication (step 2))

Recall formula for projection onto L:

Set $w := \begin{bmatrix} a \\ b \end{bmatrix}$, then $F(v) = \dfrac{w \cdot v}{w \cdot w}\, w$ ← w is a vector

$\dfrac{w \cdot v}{w \cdot w}$ ← a number

(step 1)

Let v_1 and v_2 be in $\mathbb{R}^2$.

$$F(v_1 + v_2) = \frac{w \cdot (v_1 + v_2)}{w \cdot w}\, w$$

$$= \frac{w \cdot v_1 + w \cdot v_2}{w \cdot w}\, w$$

Why is $w \cdot (v_1 + v_2) = w \cdot v_1 + w \cdot v_2$?

Answer: Let $v_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $v_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$w \cdot (v_1 + v_2) = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

$$= a(x_1 + x_2) + b(y_1 + y_2)$$

$$w \cdot v_1 + w \cdot v_2 = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$= a x_1 + b y_1 + a x_2 + b y_2$$

$$= \left(\frac{w \cdot v_1}{w \cdot w} + \frac{w \cdot v_2}{w \cdot w} \right) w$$

$$= \frac{w \cdot v_1}{w \cdot w} w + \frac{w \cdot v_2}{w \cdot w} w$$

$$= F(v_1) + F(v_2)$$

So F preserves addition. :)

(Step 2)

Let v be in $\mathbb{R}^2$ (meaning v is a vector of height 2) and c be in $\mathbb{R}$ (meaning c is a number)

$$F(cv) = \frac{w \cdot cv}{w \cdot w} w$$

$$= \frac{c(w \cdot v)}{w \cdot w} w$$

$$= c\, F(v)$$

Why is $w \cdot cv = c(w \cdot v)$?

Answer: Let $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$w \cdot cv = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} cx \\ cy \end{bmatrix} = a\,cx + b\,cy$

$c(w \cdot v) = c\left(\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}\right) = c(ax + by)$

So F preserves scalar multiplication.

So F is a linear transformation — end of 1 —

Exercise 4

Let L be the line through the points $(0,0)$ and (a,b), where $(a,b) \neq (0,0)$, and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto L.

1. Show that F is a linear transformation.
2. Find a matrix A such that $F = T_{\mathrm{A}}$.

(2) To find A so that $F = T_A$, we just need to compute $F(e_i)$ for all standard basis vectors

$$F(e_1) = \frac{w \cdot e_1}{w \cdot w} w = \frac{\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{a}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{a^2}{a^2+b^2} \\ \frac{ab}{a^2+b^2} \end{bmatrix}$$

$$F(e_2) = \frac{w \cdot e_2}{w \cdot w} w = \frac{\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{b}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{ab}{a^2+b^2} \\ \frac{b^2}{a^2+b^2} \end{bmatrix}$$

Therefore,

$$A = \begin{bmatrix} \frac{a^2}{a^2+b^2} & \frac{ab}{a^2+b^2} \\ \frac{ab}{a^2+b^2} & \frac{b^2}{a^2+b^2} \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$

where $F = T_A$

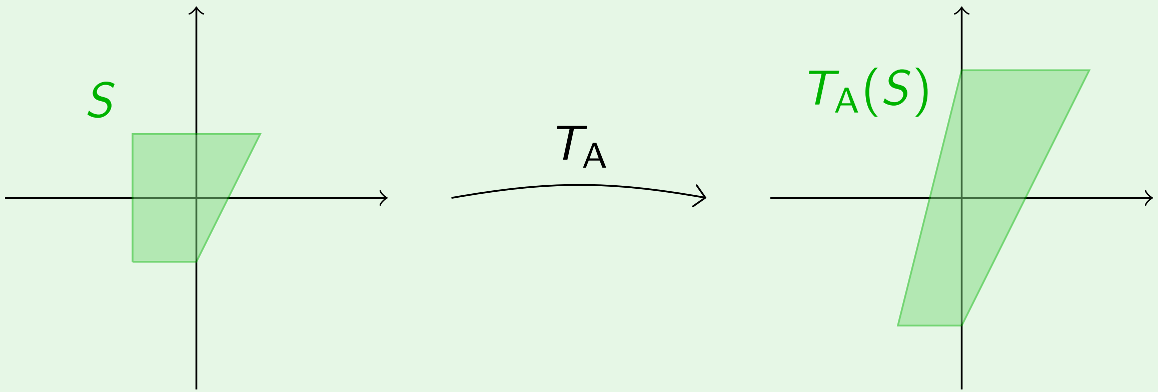
— end of (2) —

Idea: Acting on sets instead of a point

Notation: If S is a set in the domain of f, then $f(S)$ is the set of all outputs obtained by plugging in the elements of S.

Example

If $A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$, we can find what T_A does to some shape S.



It turns out functions are linear if they preserve certain shapes.

Theorem: A geometric characterization of linear transformations

A transformation $\mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if it sends...

- lines to lines,
- triangles to triangles, and
- the origin to the origin.

In practice, this is often harder to check than the previous theorem. (Theorem 1)

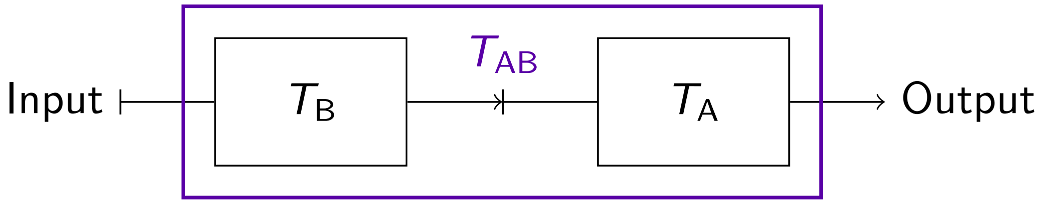
Linear transformations don't necessarily preserve other shapes!
They can send squares to parallelograms and circles to ellipses!

Matrix multiplication revisited

Matrix multiplication corresponds to composition of functions:

$$T_{AB} = T_A \circ T_B$$

That is, inputing a vector into T_{AB} is the same as first inputing it into T_B and then taking the output and plugging it into T_A.



Exercise 5

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Compute A^4 without computing any products.

Recall: Multiplication by A rotates a vector 90° counterclockwise.

The above says $T_{AAAA} = T_A \circ T_A \circ T_A \circ T_A$,

so T_{AAAA} rotates a vector 90° counterclockwise FOUR TIMES.

So T_{A^4} rotates a vector 360°, that is, fixes a vector.

So $T_{A^4}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$. So $A^4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. So $\boxed{A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$.

Check: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^4 \overset{?}{=} \mathrm{Id}$

Matrix inverses revisited

If A is invertible, then $T_{A^{-1}}$ is the function which 'undoes' T_A.

This is called the inverse function to the original function.

Examples

- If $T_M : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors by 90° counterclockwise, then $T_{M^{-1}}$ rotates vectors by 90° clockwise.
- The inverse of a reflection is itself.
- Projections are not invertible. Why? Because they cannot be undone (multiple vectors go to the same point, so information is lost).

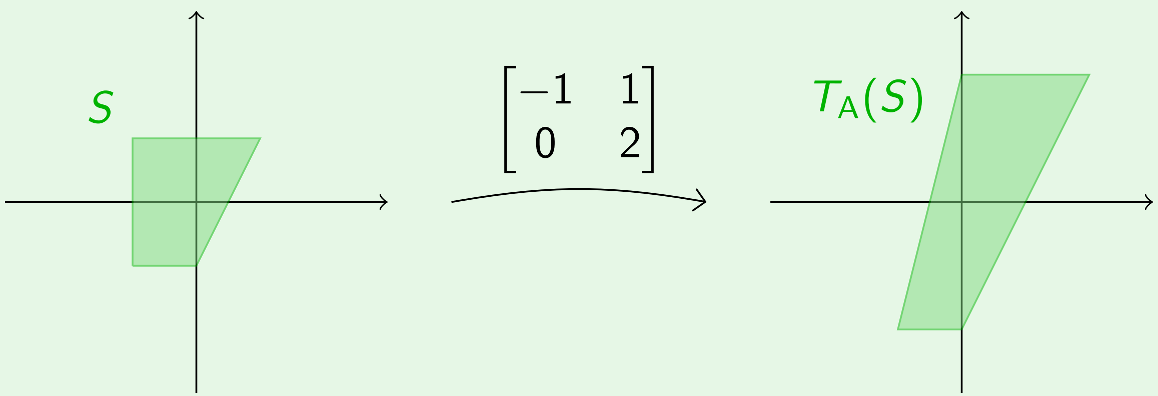
2×2 determinants revisited

If A is a 2×2 matrix and S is any shape in the plane $\mathbb{R}^2$, then

$$\mathrm{Area}(T_A(S)) = |\det(A)|\,\mathrm{Area}(S)$$

I.e. $|\det(A)|$ equals $\frac{\text{Area of output}}{\text{Area of input}}$.

Example



$\mathrm{Area}(S) = 3$ $\qquad$ $\det(A) = -2$ $\qquad$ $\mathrm{Area}(T_A(S)) = 6$

A similar result is true in 3D space.

3×3 determinants revisited

If S is a nice 3D shape, Then

$$\mathrm{Vol}(T_A(S)) = |\det(A)|\mathrm{Vol}(S)$$

Larger determinants

This can be extended to larger determinants with a notion of *n-dimensional volume* that can be defined in terms of integrals.

Recall (Def): An eigenvector of a matrix A is a non-zero vector v where $Av = cv$ for some number c.

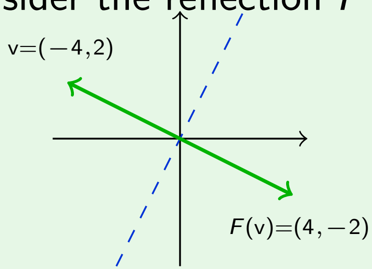
Eigenvectors revisited

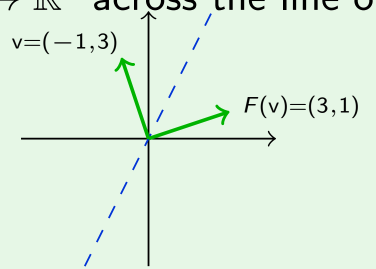
Def: An eigenvector of a linear transformation F is a non-zero vector v where $F(v)$ points in the same or opposite direction as v (equivalently, $F(v)$ is a vector parallel to v).

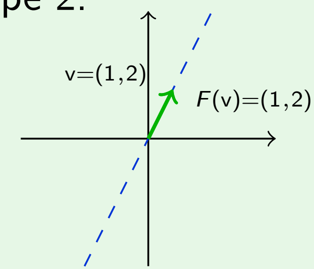
Fact: A nonzero vector v is an eigenvector of T_A if and only if v is an eigenvector of A.

Example

Consider the reflection $F : \mathbb{R}^2 \to \mathbb{R}^2$ across the line of slope 2.







$F(\mathrm{v}) = (-1)\ \mathrm{v}$ | $F(\mathrm{v})$ not parallel to v | $F(\mathrm{v}) = 1\ \mathrm{v}$

Eigenvector! | Not an eigenvector | Eigenvector!

$\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ is an eigenvector $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is not an eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector the corresponding eigenvalue is -1

the corresponding eigenvalue is 1.

Convince yourself:

The only eigenvectors of F are vectors perpendicular to the line $y = 2x$

or

vectors parallel to the line $y = 2x$