Lecture 10a

Linear Transformations

Last time

Definition: Given a matrix A, the **linear transformation of** A is the function T_A defined by left multiplication by A, that is,

$$T_{\mathrm{A}}(\mathrm{v}) := \mathrm{Av}$$

Examples of linear transformations

- Rotations
- Reflections
- Projections

A non-linear transformation

- Translation

Goal: How do we tell whether a transformation is linear (that is, comes from a matrix)?

Function terminology

A **function** in mathematics is a rule for taking in an input and returning an output. Pictorially:

$$\text{Input} \longmapsto \boxed{\text{Function}} \longrightarrow \text{Output}$$

The data defining a function also includes two sets.

- The **domain**: the set of possible inputs.
- The **target**: the set of allowed outputs.

Functions may also be called maps, operations, or transformations.

The target is also called the codomain in some textbooks

Examples

The function $f(x) = x^2 + 1$ inputs numbers and outputs numbers.

$$3 \longmapsto \boxed{\; f \;} \longrightarrow 10$$

That is, f has domain $\mathbb{R}$ and target $\mathbb{R}$.

set of all real numbers

set of all real numbers

Rotating a vector in the plane 90° counterclockwise defines a function r with domain $\mathbb{R}^2$ and target $\mathbb{R}^2$.

set of vectors of height 2

set of vectors of height 2

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \longmapsto \boxed{\; r \;} \longrightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The derivative $\frac{d}{dx}$ is an operation (i.e. a function) which inputs differentiable functions of x and outputs functions of x.

$$x^2 + 1 \longmapsto \boxed{\; \frac{d}{dx} \;} \longrightarrow 2x$$

Function notation and terminology

We can name the function and give the domain and target as:

$$F : A \longrightarrow B$$

the function · the domain · the target

This would be read aloud as '*F* from *A* to *B*'.

Examples

- Consider a function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$.
- Let $r : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by 90° clockwise.
- Differentiation by x is a function $\frac{d}{dx} : \underbrace{C^1(x)}_{\text{set of differentiable functions}} \to \underbrace{C^0(x)}_{\text{set of continuous functions}}$.

Recall: $\mathbb{R}^d = \{\text{vectors of height } d\}$

Domain and target of linear transformation

An $m \times n$-matrix A gives a linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

domain of T_A is the set of vectors of height n

target of T_A is the set of vectors of height m

- The **domain** of T_A is $\mathbb{R}^n$
  That is, the function T_A inputs vectors of height n
- The **target** of T_A is $\mathbb{R}^m$
  That is, the function T_A outputs vectors of height m
- The name of the function is T_A. This special notation that reminds us that the function is given by multiplication by the matrix A.

  That is, T_A can only input vectors whose height is $\mathrm{width}(A)$, and outputs vectors whose height is $\mathrm{height}(A)$.

Restating the problem

Given a function $F : \mathbb{R}^n \to \mathbb{R}^m$, when is there an $m \times n$-matrix A such that $F = T_{\mathrm{A}}$?

Plan: Find nice properties that characterize linear transformations.

One of the properties of linear transformations

Linear transformations send zero vectors to zero vectors.

Why? Multiplication by a zero vector gives a zero vector.

Last time: The function that translates a point in $\mathbb{R}^2$ to the right by 1

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y \end{bmatrix}$$

cannot be a linear transformation. Why not? Note that

$$F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so F sends the zero vector to a non-zero vector.

Properties of linear transformations

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

- T preserves addition. If v and w are in $\mathbb{R}^n$, then

$$T(\mathrm{v} + \mathrm{w}) = T(\mathrm{v}) + T(\mathrm{w})$$

- T preserves scalar multiplication. If v is in $\mathbb{R}^n$ and c is in $\mathbb{R}$, then

$$T(c\mathrm{v}) = cT(\mathrm{v})$$

E.g. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$. Then ...

"T preserves addition" $T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) = T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right)$ for all a, b, c, d in $\mathbb{R}$

"T preserves scalar multiplication" $T\left(k\begin{bmatrix} a \\ b \end{bmatrix}\right) = k\,T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$ for all a, b, k in $\mathbb{R}$

Each follows directly from a property of matrix multiplication.

$$T_A(\mathrm{v} + \mathrm{w}) \overset{\text{def of } T_A}{=} A(\mathrm{v} + \mathrm{w}) \underset{\text{distributivity}}{=} A\mathrm{v} + A\mathrm{w} \overset{\text{def of } T_A}{=} T_A(\mathrm{v}) + T_A(\mathrm{w})$$

$$T_A(c\mathrm{v}) \overset{\text{def of } T_A}{=} A(c\mathrm{v}) \underset{\text{matrix arithmetic}}{=} c\,A\mathrm{v} \overset{\text{def of } T_A}{=} c\,T_A(\mathrm{v})$$

Exercise 1

Show that the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ x + y \end{bmatrix}$$

is not a linear transformation.

Strategy for showing that a function F is <u>not</u> a linear transformation.

- Check $F(\vec{0})$. If $F(\vec{0}) \neq \vec{0}$, then you are done.
- Try v, w and check $F(v+w) \neq F(v)+F(w)$.
- Try v and a number $c \neq 1$.
  Check $F(cv) \neq c\,F(v)$.

Exercise 1

Show that the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ x+y \end{bmatrix}$$

is not a linear transformation.

- ~~Check $F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (not helpful)~~

Don't include scratch work

Answer to Exercise 1:

- Try $v := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $w := \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$F(v+w) = F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = F\left(\begin{bmatrix} 1+3 \\ 2+4 \end{bmatrix}\right) = F\left(\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 4^2 \\ 4+6 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

$$F(v)+F(w) = F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + F\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1^2 \\ 1+2 \end{bmatrix} + \begin{bmatrix} 3^2 \\ 3+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$F(v+w) \neq F(v) + F(w)$$

So F does not preserve addition,
so F is <u>not</u> a linear transformation.

We can combine the two rules above into a single rule.

Properties of linear transformations, restated

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T preserves linear combinations. Meaning,

if $v_1, v_2, \ldots, v_k$ are in $\mathbb{R}^n$ and $c_1, c_2, \ldots, c_k$ are in $\mathbb{R}$, then

$$T(c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_k T(v_k)$$

E.g. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then

$$T\left(c_1 \begin{bmatrix} a \\ b \end{bmatrix} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} + c_3 \begin{bmatrix} e \\ f \end{bmatrix}\right) =$$

$$c_1 T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + c_2 T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) + c_3 T\left(\begin{bmatrix} e \\ f \end{bmatrix}\right)$$

for all $c_1, c_2, c_3, a, b, c, d, e, f$ in $\mathbb{R}$

Exercise 2

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and assume we know

$$T\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = \begin{bmatrix}3\\-2\end{bmatrix} \text{ and } T\left(\begin{bmatrix}2\\4\end{bmatrix}\right) = \begin{bmatrix}-3\\4\end{bmatrix}$$

Find $T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$.

Strategy:
- Step 1: Write $\begin{bmatrix}-1\\1\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$
- Step 2: Use the property: T preserves linear combinations.

Step 1

$$\begin{bmatrix}-1\\1\end{bmatrix} = c_1\begin{bmatrix}1\\3\end{bmatrix} + c_2\begin{bmatrix}2\\4\end{bmatrix}$$

$$\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}c_1 + 2c_2\\3c_1 + 4c_2\end{bmatrix}$$

$$c_1 + 2c_2 = -1$$
$$3c_1 + 4c_2 = 1$$

This is a system of two linear equations in c_1, c_2 equivalent to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 3 & 4 & 1 \end{array}\right]$$

$$R_2 \mapsto -3R_1 + R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -2 & 4 \end{array}\right]$$

$$R_2 \mapsto -\tfrac{1}{2}R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -2 \end{array}\right] \qquad \boxed{c_2 = -2}$$

$$c_1 + 2c_2 = -1 \Rightarrow c_1 + 2(-2) = -1$$
$$\Rightarrow c_1 - 4 = -1$$
$$\Rightarrow \boxed{c_1 = 3}$$

So $\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + -2\begin{bmatrix} 2 \\ 4 \end{bmatrix}}$ ← answer to step 1

Check: $-1 = 3.1 - 2.2$ ✓
$1 = 3.3 - 2.4$ ✓

Step 2

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = T\left(3\begin{bmatrix} 1 \\ 3 \end{bmatrix} + -2\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

T preserves linear combinations because T is a linear transformations

$$= 3\,\underbrace{T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)} + -2\,\underbrace{T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)}$$

$$= \underbrace{3\begin{bmatrix} 3 \\ -2 \end{bmatrix}} - 2\begin{bmatrix} -3 \\ 4 \end{bmatrix} \quad \left(\text{because } T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 9 \\ -6 \end{bmatrix} + \begin{bmatrix} 6 \\ -8 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 15 \\ -14 \end{bmatrix}}$$

~ the end of Exercise 2 ~

Exercise 2 [(1) T is a lin. transformation] implies [(3) T preserves linear combination]

Exercise 1 If [not (2): T doesn't preserve addition OR T doesn't preserve scalar multiplication]
then [not (1): T is not a linear transformation]

Theorem 1

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$. Then the following three statements are equivalent.

1. $T = T_A$ for some $m \times n$ matrix A; that is, T is a linear transformation.
2. T preserves addition and scalar multiplication.
3. T preserves linear combinations.

Next time: a trick for computing the above matrix.