

Lecture 10a

Linear Transformations

Last time

Definition: Given a matrix A , the **linear transformation of A** is the function T_A defined by left multiplication by A , that is,

$$T_A(v) := Av$$

Examples of linear transformations

- Rotations
- Reflections
- Projections

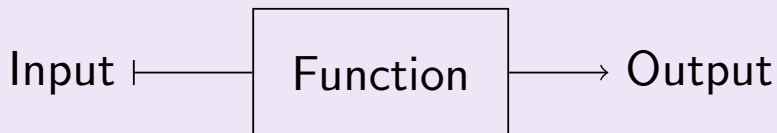
A non-linear transformation

- Translation

Goal: How do we tell whether a transformation is linear (that is, comes from a matrix)?

Function terminology

A **function** in mathematics is a rule for taking in an input and returning an output. Pictorially:



The data defining a function also includes two sets.

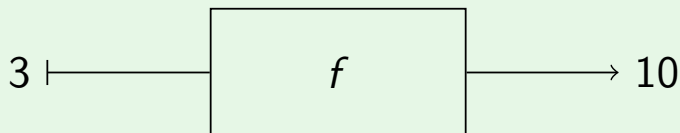
- The **domain**: the set of possible inputs.
- The **target**: the set of allowed outputs.

Functions may also be called **maps**, **operations**, or **transformations**.

The **target** is also called the **codomain** in some textbooks

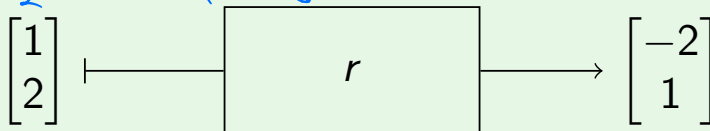
Examples

The function $f(x) = x^2 + 1$ inputs numbers and outputs numbers.

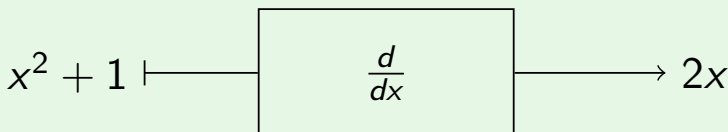


That is, f has domain \mathbb{R} and target \mathbb{R} .
set of all real numbers *set of all real numbers*

Rotating a vector in the plane 90° counterclockwise defines a function r with domain \mathbb{R}^2 and target \mathbb{R}^2 .
set of vectors of height 2 *set of vectors of height 2*



The derivative $\frac{d}{dx}$ is an operation (i.e. a function) which inputs differentiable functions of x and outputs functions of x .



Function notation and terminology

We can name the function and give the domain and target as:

$$\begin{array}{ccccc} & F : A & \longrightarrow & B & \\ \nearrow & & & & \nearrow \\ \text{the function} & & \text{the domain} & & \text{the target} \end{array}$$

This would be read aloud as '*F* from *A* to *B*'.

Examples

▶ Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2 + 1$.

▶ Let $r : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be rotation by 90° clockwise.

▶ Differentiation by x is a function $\frac{d}{dx} : \underbrace{C^1(x)}_{\text{set of differentiable functions}} \rightarrow \underbrace{C^0(x)}_{\text{set of continuous functions}}$.

Recall: $\mathbb{R}^d = \{\text{vectors of height } d\}$

Domain and target of linear transformation

An $m \times n$ -matrix A gives a linear transformation $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

domain of T_A is
the set of vectors
of height n

target of T_A
is the set of
vectors of
height m

- ▶ The **domain** of T_A is \mathbb{R}^n
That is, the function T_A inputs vectors of height n
- ▶ The **target** of T_A is \mathbb{R}^m
That is, the function T_A outputs vectors of height m
- ▶ The name of the function is T_A . This special notation that reminds us that the function is given by multiplication by the matrix A .
That is, T_A can only input vectors whose height is $\text{width}(A)$, and outputs vectors whose height is $\text{height}(A)$.

Restating the problem

Given a function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$, when is there an $m \times n$ -matrix A such that $F = T_A$?

Plan: Find nice properties that characterize linear transformations.

One of the properties of linear transformations

Linear transformations send zero vectors to zero vectors.

Why? Multiplication by a zero vector gives a zero vector.

Last time: The function that translates a point in \mathbb{R}^2 to the right by 1

$$F \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x + 1 \\ y \end{bmatrix}$$

cannot be a linear transformation. Why not? Note that

$$F \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so F sends the zero vector to a non-zero vector.

Properties of linear transformations

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation.

- T preserves addition. If v and w are in \mathbb{R}^n , then

$$T(v + w) = T(v) + T(w)$$

- T preserves scalar multiplication. If v is in \mathbb{R}^n and c is in \mathbb{R} , then

$$T(cv) = cT(v)$$

E.g. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Then ...

"T preserves addition"

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) = T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \text{ for all } a, b, c, d \text{ in } \mathbb{R}$$

"T preserves scalar multiplication"

$$T\left(k \begin{bmatrix} a \\ b \end{bmatrix}\right) = k T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \text{ for all } a, b, k \text{ in } \mathbb{R}$$

Each follows directly from a property of matrix multiplication.

$$T_A(v + w) \stackrel{\text{def of } T_A}{=} A(v + w) \stackrel{\text{distributivity}}{=} Av + Aw \stackrel{\text{def of } T_A}{=} T_A(v) + T_A(w)$$

$$T_A(cv) \stackrel{\text{def of } T_A}{=} A(cv) \stackrel{\text{matrix arithmetic}}{=} c Av \stackrel{\text{def of } T_A}{=} c T_A(v)$$

Exercise 1

Show that the function $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$F \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x^2 \\ x + y \end{bmatrix}$$

is not a linear transformation.

Strategy for showing that a function F is not a linear transformation.

- Check $F(\vec{0})$. If $F(\vec{0}) \neq \vec{0}$, then you are done.
- Try v, w and check $F(v+w) \neq F(v)+F(w)$.
- Try v and a number $c \neq 1$.
Check $F(cv) \neq cF(v)$.

Exercise 1

Show that the function $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ x+y \end{bmatrix}$$

is not a linear transformation.

- Check ~~$F\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$~~ (not helpful) *Don't include scratch work*

Answer to Exercise 1:

- Try $v := \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $w := \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\begin{aligned} F(v+w) &= F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) \\ &= F\left(\begin{bmatrix} 1+3 \\ 2+4 \end{bmatrix}\right) \\ &= F\left(\begin{bmatrix} 4 \\ 6 \end{bmatrix}\right) \\ &= \begin{bmatrix} 4^2 \\ 4+6 \end{bmatrix} \\ &= \begin{bmatrix} 16 \\ 10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} F(v) + F(w) &= F\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + F\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1^2 \\ 1+2 \end{bmatrix} + \begin{bmatrix} 3^2 \\ 3+4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 9 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 10 \\ 10 \end{bmatrix} \end{aligned}$$

$$F(v+w) \neq F(v) + F(w)$$

So F does not preserve addition,
so F is not a linear transformation.

We can combine the two rules above into a single rule.

Properties of linear transformations, restated

If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation, then T **preserves linear combinations**. Meaning,

if v_1, v_2, \dots, v_k are in \mathbb{R}^n and c_1, c_2, \dots, c_k are in \mathbb{R} , then

$$T(c_1v_1 + c_2v_2 + \cdots + c_kv_k) = c_1T(v_1) + c_2T(v_2) + \cdots + c_kT(v_k)$$

E.g. If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation, then

$$T\left(c_1 \begin{bmatrix} a \\ b \end{bmatrix} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} + c_3 \begin{bmatrix} e \\ f \end{bmatrix}\right) =$$

$$c_1 T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + c_2 T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) + c_3 T\left(\begin{bmatrix} e \\ f \end{bmatrix}\right)$$

for all $c_1, c_2, c_3, a, b, c, d, e, f$ in \mathbb{R}

Exercise 2

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation, and assume we know

$$T \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \text{and} \quad T \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Find $T \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$.

Strategy: • Step 1 Write $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$
• Step 2 Use the property: T preserves linear combinations.

Step 1

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 \\ 3c_1 + 4c_2 \end{bmatrix}$$

$$c_1 + 2c_2 = -1$$

$$3c_1 + 4c_2 = 1$$

This is a system of two linear equations in C_1, C_2 equivalent to the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & -1 \\ 3 & 4 & 1 \end{array} \right]$$

$$R_2 \mapsto -3R_1 + R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -2 & 4 \end{array} \right]$$

$$R_2 \mapsto -\frac{1}{2}R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -2 \end{array} \right] \quad \boxed{C_2 = -2}$$

$$C_1 + 2C_2 = -1 \Rightarrow C_1 + 2(-2) = -1$$

$$\Rightarrow C_1 - 4 = -1$$

$$\Rightarrow \boxed{C_1 = 3}$$

$$\text{So } \boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + -2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

← answer to step 1

Check: $-1 = 3 \cdot 1 - 2 \cdot 2 \quad \checkmark$
 $1 = 3 \cdot 3 - 2 \cdot 4 \quad \checkmark$

Step 2

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = T\left(3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + -2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

T preserves linear combinations because T is a linear transformation

$$= 3 T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) + -2 T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right)$$

$$= 3 \begin{bmatrix} 3 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} \quad \left(\begin{array}{c} \text{because} \\ T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \end{array} \right)$$

$$= \begin{bmatrix} 9 \\ -6 \end{bmatrix} + \begin{bmatrix} 6 \\ -8 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 15 \\ -14 \end{bmatrix}}$$

~ the end of Exercise 2 ~

Exercise 2 [① T is a lin. transformation] implies [③ T preserves linear combinations]

Exercise 1 If [not ②: T doesn't preserve addition OR T doesn't preserve scalar multiplication] then [not ①: T is not a linear transformation]

Theorem 1

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Then the following three statements are equivalent.

- ① $T = T_A$ for some $m \times n$ matrix A ; that is, T is a linear transformation.
- ② T preserves addition and scalar multiplication.
- ③ T preserves linear combinations.

Next time: a trick for computing the above matrix.