# Linear Recurrences

## Example

The **Fibonacci Numbers** are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

and can be defined by the linear recurrence relation

$$f_{n+2} = f_{n+1} + f_n \text{ for all } n \geq 0,$$

with the initial conditions $f_0 = 1$ and $f_1 = 1$.

## Problem

*Find* $f_{100}$.

Instead of using the recurrence to compute $f_{100}$, we'd like to find a formula for $f_n$ that holds for all $n \geq 0$.

(Note: For simplicity, in this document we will work with other recurrences, corresponding to matrices with integer/fraction eigenvalues.)

A linear recurrence of length $k$ has the form

$$x_{n+k} = a_1 x_{n+k-1} + a_2 x_{n+k-2} + \cdots + a_k x_n,\ n \geq 0,$$

for some real numbers $a_1, a_2, \ldots, a_k$.

## Example

The simplest linear recurrence has length one, so has the form

$$x_{n+1} = a x_n \text{ for } n \geq 0,$$

with $a \in \mathbb{R}$ and some initial value $x_0$.

**Solution.** In this case,

$$\begin{aligned} x_1 &= a x_0 \\ x_2 &= a x_1 = a^2 x_0 \\ x_3 &= a x_2 = a^3 x_0 \\ \vdots & \quad \vdots \quad \vdots \\ x_n &= a x_{n-1} = a^n x_0 \end{aligned}$$

Therefore, $x_n = a^n x_0$.

## Example

Let $x_0 = 0$ and $x_1 = 1$, and $x_{n+2} = 2x_{n+1} + 3x_n$ for $n \geq 0$. Find a formula for $x_n$.

**Solution.** Define $V_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ for each $n \geq 0$. Then

$$V_0 = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and for $n \geq 0$,

$$V_{n+1} = \begin{bmatrix} x_{n+1} \\ x_{n+2} \end{bmatrix} = \begin{bmatrix} x_{n+1} \\ 2x_{n+1} + 3x_n \end{bmatrix}$$

Now express $V_{n+1} = \begin{bmatrix} x_{n+1} \\ 2x_{n+1} + 3x_n \end{bmatrix}$ as a matrix product:

$$V_{n+1} = \begin{bmatrix} 0 + x_{n+1} \\ 3x_n + 2x_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} = AV_n$$

This is a linear dynamical system, so we can apply the techniques from individual project: linear dynamical system (click here), provided that $A$ is diagonalizable.

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x & -1 \\ -3 & x-2 \end{vmatrix} = x^2 - 2x - 3 = (x-3)(x+1)$$

Therefore $A$ has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$, and is diagonalizable.

## Example (continued)

$\mathrm{x}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 3$, and $\mathrm{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = -1$.

Since $A$ has distinct eigenvalues, we know it's diagonalizable. Let $P := \begin{bmatrix} \mathrm{x}_1 & \mathrm{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$. Then $A = PDP^{-1}$

Writing $P^{-1}V_0 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, we get

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Therefore,

$$\begin{aligned} V_n = \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix} &= b_1\lambda_1^n\mathrm{x}_1 + b_2\lambda_2^n\mathrm{x}_2 \\ &= \frac{1}{4}3^n\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{1}{4}(-1)^n\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \end{aligned}$$

and so

$$\boxed{x_n = \frac{1}{4}3^n - \frac{1}{4}(-1)^n}.$$

## Example

Solve the recurrence relation $x_{k+2} = 5x_{k+1} - 6x_k$, $k \geq 0$ with $x_0 = 0$ and $x_1 = 1$.

**Solution.** Write

$$V_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ -6x_k + 5x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$$

Find the eigenvalues and corresponding eigenvectors for $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$: $A$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ with eigenvectors $\mathrm{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathrm{x}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, P^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1} V_0 = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Finally,

$$V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = b_1 \lambda_1^k \mathrm{x}_1 + b_2 \lambda_2^k \mathrm{x}_2 = (-1)2^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3^k \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = (-1)2^k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3^k \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and therefore $\boxed{x_k = 3^k - 2^k}$.

# Student Individual Final PROBLEM 1

Let $x_0 = 0$ and $x_1 = 1$, and $x_{n+2} = x_{n+1} + 2x_n$ for $n \geq 0$. We will go through the steps to find a formula for $x_n$ following the above methods (via diagonalization).

Define $V_n := \begin{bmatrix} x_n \\ x_{n+1} \end{bmatrix}$ for each $n \geq 0$. So $V_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

a. Write down the matrix $A$ where $V_{n+1} = AV_n$.
b. By hand, compute the characteristic polynomial of $A$. Use it to compute the eigenvalues of $A$.
c. Write down $D$, a diagonal matrix whose entries are the eigenvalues you computed. Put the larger eigenvalue $\lambda$ on the first column of $D$. Write down an invertible $P$ so that $A = PDP^{-1}$. (There are different ways to do this. Please use the algorithm covered in class lectures www.youtube.com/channel/UC2UZ2jPm5y7T2rvLtZY9llg)
d. Compute $P^{-1}$ by hand using the row reduce algorithm, but check with a software afterwards.
e. Use the eigenvalues, $P$, and $P^{-1}$ to write down a formula for $x_n$, as shown above.

# Student Individual Final PROBLEM 2

(See the student solution manual's answer for Exercise 3.4.2(b) for a similar example.) Let $x_0 = 1$, $x_1 = 0$, $x_2 = 1$, and $x_{n+3} = 6x_{n+2} - 11x_{n+1} + 6x_n$ for $n \geq 0$. We will go through the steps to find a formula for $x_n$ following the above methods (via diagonalization).

Define $V_n := \begin{bmatrix} x_n \\ x_{n+1} \\ x_{n+2} \end{bmatrix}$ for each $n \geq 0$. So $V_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

- a) Write down a matrix $B$ where $V_{n+1} = BV_n$. (I have computed for you that this matrix has three eigenvalues: $3, 2, 1$.)
- b) Using a software (or by hand), find three eigenvectors corresponding to the three eigenvalues. Then find write down an eigenbasis for $B$.
- c) Let $D := \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Use the eigenbasis to write down an invertible matrix $P$ such that $B = PDP^{-1}$. (There are different ways to do this. Please use the algorithm covered in class lectures www.youtube.com/channel/UC2UZ2jPm5y7T2rvLtZY9llg.)
- d) Compute $P^{-1}$ using the row reduce algorithm. Use a software to do the row reduce (or do by hand). Write down your input and output.
- e) Use $P^{-1}$, $P$ and the eigenvalues to write down a formula for $x_n$, as shown above (or as in Exercise 3.4.2(b) in the solution manual).