Example

Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = \frac{3}{2}$.

Solution. We want to find the coefficients r_0, r_1, r_2 and r_3 of

$$p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

so that $p(0) = 1$, $p(1) = 2$, $p(2) = 5$, and $p(3) = 10$.

$$\begin{array}{rcl} p(0) & = & r_0 = 1 \\ p(1) & = & r_0 + r_1 + r_2 + r_3 = 2 \\ p(2) & = & r_0 + 2r_1 + 4r_2 + 8r_3 = 5 \\ p(3) & = & r_0 + 3r_1 + 9r_2 + 27r_3 = 10 \end{array}$$

Solve this system of four equations in the four variables r_0, r_1, r_2 and r_3.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore $r_0 = 1$, $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, and so $\boxed{p(x) = 1 + x^2}$.

$$\text{The estimate is } y = p\left(\frac{3}{2}\right) = 1 + \left(\frac{3}{2}\right)^2 = \boxed{\frac{13}{4}}.$$

Theorem (§3.2 Theorem 6)

Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

The polynomial $p(x)$ is called the interpolating polynomial for the data.

To find $p(x)$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \ldots, r_{n-1}$. $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$:

$$\begin{array}{rcl} r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{n-1} x_1^{n-1} & = & y_1 \\ r_0 + r_1 x_2 + r_2 x_2^2 + \cdots + r_{n-1} x_2^{n-1} & = & y_2 \\ r_0 + r_1 x_3 + r_2 x_3^2 + \cdots + r_{n-1} x_3^{n-1} & = & y_3 \\ \vdots & \vdots & \vdots \\ r_0 + r_1 x_n + r_2 x_n^2 + \cdots + r_{n-1} x_n^{n-1} & = & y_n \end{array}$$

The coefficient matrix for this system is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

The determinant of a matrix of this form is called a Vandermonde determinant.

The Vandermonde Determinant

Theorem (§3.2 Theorem 7)

Let $a_1, a_2, \ldots, a_n$ be real numbers, $n \geq 2$. The the corresponding Vandermonde determinant is

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} = \Pi_{1 \leq j < i \leq n}(a_i - a_j).$$

For example, see the next slide.

Example

In our earlier example with the data points $(0,1)$, $(1,2)$, $(2,5)$ and $(3,10)$, we have

$$a_1 = 0, a_2 = 1, a_3 = 2, a_4 = 3$$

giving us the Vandermonde determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

According to Theorem 7, this determinant is equal to

$$\begin{aligned} & (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \\ = \quad & (1-0)(2-0)(2-1)(3-0)(3-1)(3-2) = 2 \times 3 \times 2 = 12. \end{aligned}$$

By Theorem 7, the Vandermonde determinant is nonzero if $a_1, a_2, \ldots, a_n$ are distinct.

This means that given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with distinct x_i, then there is a unique interpolating polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}.$$

Student Individual Final PROBLEM 1

Explain each step in your computation. It's OK to use a calculator to row reduce (or to check your answers).

- a. Consider these four data points: $(1, 2)$, $(2, 5)$, $(3, 10)$, and $(4, -1)$.
  - a. Find the interpolating polynomial $g(x)$ in $\mathbb{P}_3$ using the methods shown above. (note: we've shown above that this polynomial exists and is unique, and recall from Lecture 16–17 in youtube.com/channel/UC2UZ2jPm5y7T2rvLtZY9llg that $\mathbb{P}_3$ is the set of polynomials of degree **at most** 3).
  - b. Use $g(x)$ to estimate the value of y corresponding to $x = 0$.

- b. A forest manager wants to estimate the age (in decades) of a tree by measuring the diameter of the trunk (in meters). She obtains the following table of data:

	Tree 1	Tree 2	Tree 3
Trunk Diameter	1	2	3
Age	6	10	12

  - a. Compute a polynomial $f(x)$ in $\mathbb{P}_2$ which fits this data using the methods shown above. (note: we've shown above that this polynomial exists and is unique, and recall that $\mathbb{P}_2$ is the set of polynomials of degree **at most** 2).
  - b. Use $f(x)$ to estimate the age of a tree with a trunk diameter of 1.5 m.