Markov Chains

Markov Chains are used to model systems (or processes) that evolve through a series of stages. At each stage, the system is in one of a finite number of states.

Example (Weather Model)

Three states: sunny (S), cloudy (C), rainy (R).
Stages: days.

The state that the system occupies at any stage is determined by a set of probabilities.
Important fact: probabilities are always real numbers between zero and one, inclusive.

Example (Weather Model – continued)

- If it is sunny one day, then there is a 40% chance it will be sunny the next day, and a 40% chance that it will be cloudy the next day (and a 20% chance it will be rainy the next day).

The values 40%, 40% 20% are transition probabilities, and are assumed to be known.

- If it is cloudy one day, then there is a 40% chance it will be rainy the next day, and a 25% chance that it will be sunny the next day.
- If it is rainy one day, then there is a 30% chance it will be rainy the next day, and a 50% chance that it will be cloudy the next day.

Example (Weather Model – continued)

We put the transition probabilities into a transition matrix,

$$P = \begin{bmatrix} 0.4 & 0.25 & 0.2 \\ 0.4 & 0.35 & 0.5 \\ 0.2 & 0.4 & 0.3 \end{bmatrix}$$

Note. Transition matrices are stochastic, meaning that the sum of the entries in each column is equal to 1.

Suppose that it is rainy on Thursday. What is the probability that it will be sunny on Sunday?

The initial state vector, S_0, corresponds to the state of the weather on Thursday, so

$$S_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example (Weather Model – continued)

What is the state vector for Friday?

$$S_1 = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.25 & 0.2 \\ 0.4 & 0.35 & 0.5 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = PS_0.$$

To find the state vector for Saturday:

$$S_2 = PS_1 = \begin{bmatrix} 0.4 & 0.25 & 0.2 \\ 0.4 & 0.35 & 0.5 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.265 \\ 0.405 \\ 0.33 \end{bmatrix}$$

Finally, the state vector for Sunday is

$$S_3 = PS_2 = \begin{bmatrix} 0.4 & 0.25 & 0.2 \\ 0.4 & 0.35 & 0.5 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.265 \\ 0.405 \\ 0.33 \end{bmatrix} = \begin{bmatrix} 0.27325 \\ 0.41275 \\ 0.314 \end{bmatrix}$$

The probability that it will be sunny on Sunday is 27.325%.

Important fact: the sum of the entries of a state vector is always 1.

Theorem (Theorem 1, taken from the book §2.9 Theorem 1)

If P is the transition matrix for an n-state Markov chain, then

$$S_{m+1} = PS_m \text{ for } m = 0, 1, 2, \ldots$$

Example (§2.9 Example 1)

- A customer always eats lunch either at restaurant A or restaurant B.
- The customer never eats at A two days in a row.
- If the customer eats at B one day, then the next day she is three times as likely to eat at B as at A.

What is the probability transition matrix? Answer: $P = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix}$

Initially, the customer is equally likely to eat at either restaurant, so

$$S_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, S_1 = \begin{bmatrix} 0.125 \\ 0.875 \end{bmatrix}, S_2 = \begin{bmatrix} 0.21875 \\ 0.78125 \end{bmatrix}, S_3 = \begin{bmatrix} 0.1953125 \\ 0.8046875 \end{bmatrix},$$

$$S_4 = \begin{bmatrix} 0.20117 \\ 0.79883 \end{bmatrix}, S_5 = \begin{bmatrix} 0.19971 \\ 0.80029 \end{bmatrix}, S_6 = \begin{bmatrix} 0.20007 \\ 0.79993 \end{bmatrix}, S_7 = \begin{bmatrix} 0.19998 \\ 0.80002 \end{bmatrix},$$

are calculated, and these appear to converge to $\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$.

Example (§2.9 Example 3)

A wolf pack always hunts in one of three regions, R_1, R_2, and R_3.

- If it hunts in a region one day, it is as likely as not to hunt there again the next day.
- If it hunts in R_1, it never hunts in R_2 the next day.
- If it hunts in R_2 or R_3, it is equally likely to hunt in each of the other two regions the next day.

If the pack hunts in R_1 on Monday, find the probability that it will hunt in R_3 on Friday.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We want to find S_4, and, in particular, **the last entry in S_4.**

$$S_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}, \quad S_2 = PS_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{1}{2} \end{bmatrix},$$

$$S_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{32} \\ \frac{3}{16} \\ \frac{15}{32} \end{bmatrix}, \quad S_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{11}{32} \\ \frac{3}{16} \\ \frac{15}{32} \end{bmatrix} = \begin{bmatrix} \\ \\ \frac{29}{64} \end{bmatrix}$$

Therefore, the probability of the pack hunting in R_3 on Friday is $\boxed{\frac{29}{64}}$.

Sometimes, state vectors converge to a particular vector, called the steady state vector.

Problem

How do we know if a Markov chain has a steady state vector? If the Markov chain has a steady state vector, how do we find it?

One condition ensuring that a steady state vector exists is that the transition matrix P be regular, meaning that for some integer $k > 0$, all entries of P^k are positive (i.e., greater than zero).

Example

In §2.9 Example 1, $P = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix}$ is regular because

$$P^2 = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{16} \\ \frac{3}{4} & \frac{3}{16} \end{bmatrix}$$

has all entries greater than zero.

Theorem (Theorem 2)

If P is the transition matrix of a Markov chain and P is regular, then the steady state vector can be found by solving the system

$$S = PS$$

for S, and then ensuring that the entries of S sum to one.

Notice that if $S = PS$, then

$$\begin{aligned} S - PS &= 0 \\ Id\ S - PS &= 0 \\ (Id - P)S &= 0 \end{aligned}$$

- This last line represents a system of linear equations that is homogeneous.
- This system has infinitely many solutions for any transition matrix P.
- Choose the value of the parameter so that the entries of S sum to one.

Example

$$\text{From §2.9 Example 1, } P = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix},$$

and we've already verified that P is regular.

Now solve the system $(Id - P)S = 0$.

$$Id - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{4} \\ -1 & \frac{1}{4} \end{bmatrix}$$

Solving $(Id - P)S = 0$: $\left[\begin{array}{rr|r} 1 & -\frac{1}{4} & 0 \\ -1 & \frac{1}{4} & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$

The general solution in parametric form is $s_1 = \frac{1}{4}t, s_2 = t$ for $t \in \mathbb{R}$. Since $s_1 + s_2 = 1$,

$$\begin{aligned} \frac{1}{4}t + t &= 1 \\ \frac{5}{4}t &= 1 \\ t &= \frac{4}{5} \end{aligned}$$

$$\text{Therefore, the steady state vector is } S = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}.$$

Example (§2.9 Example 3)

Is there a steady state vector? If so, find it.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}, \text{ so } P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & \frac{5}{16} \\ \frac{1}{8} & \frac{5}{16} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{7}{16} \end{bmatrix}$$

So P is regular and a steady state vector exists. Now solve the system $(Id - P)S = 0$:

$$\left[\begin{array}{ccc|c} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{4} & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right.$$

The general solution in parametric form is

$$s_3 = t, s_2 = \frac{1}{2}t, s_1 = \frac{3}{4}t, \text{ where } t \text{ is in } \mathbb{R}.$$

Since $s_1 + s_2 + s_3 = 1$,

$$t + \frac{1}{2}t + \frac{3}{4}t = 1,$$

implying that $t = \frac{4}{9}$. Therefore the steady state vector is $S = \begin{bmatrix} \frac{3}{9} \\ \frac{2}{9} \\ \frac{4}{9} \end{bmatrix}$.

Student Individual Final PROBLEM

Assume there are 3 economic classes, upper, middle, and lower, and that economic mobility behaves as follows.

- Of the children of upper-class parents, 80 percent remains upper-class, 10 percent become middle-class and 10 percent become lower-class.
- Of the children of middle-class parents, 10 percent become upper-class, 70 percent remain middle-class, and 20 percent become lower-class.
- Of the children of lower-class parents, 10 percent become upper-class, 20 percent become middle-class, and 70 percent remain lower-class

a. Write down the transition matrix P following the convention given in above slides. (Sanity check: make sure that each column of your matrix sum up to 1.)

b. What is the probability that a grandchild of lower-class parents becomes upper-class? (Hint: Use above methods. If the 3rd column of your transition matrix P correspond to lower-class parents and if you want to use an *initial state* vector S_0, you should use $S_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.)

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Student Individual Final PROBLEM (con't)

c. Determine whether P is regular (explain why or why not).

d. If P is regular, then find the steady state vector using Theorem 2 in egunawan.github.io/la/finalproject/project_markov_chains.pdf.

e. Use your previous answers to find the long-term breakdown of society into these 3 classes. If not possible, explain why.

   Hint: This is meant to be a straightforward problem. The techniques used in above examples (and examples in Section 2.9 of the textbook) are sufficient to answer this!