Linear Dynamical Systems

A **linear dynamical system** consists of

- an $n \times n$ matrix A and an n-vector V_0;
- a matrix recursion defining $V_1, V_2, V_3, \ldots$ by $V_{k+1} = AV_k$; i.e.,

$$\begin{aligned} V_1 &= AV_0 \\ V_2 &= AV_1 = A(AV_0) = A^2 V_0 \\ V_3 &= AV_2 = A(A^2 V_0) = A^3 V_0 \\ \vdots & \quad \vdots \quad \vdots \\ V_k &= A^k V_0. \end{aligned}$$

Linear dynamical systems are used, e.g., to model the evolution of populations over time.
If A is diagonalizable, then

$$A = PDP^{-1}$$

where D is a diagonal matrix with eigenvalues as entries and P is the appropriate concatenation of an eigenbasis of A.
Thus $A^k = PD^k P^{-1}$. Therefore,

$$\boxed{V_k = A^k V_0 = PD^k P^{-1} V_0}.$$

Example

Consider the linear dynamical system $V_{k+1} = AV_k$ with

$$A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}, \text{ and } V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find a formula for V_k.

Solution. First, compute the characteristic polynomial: $\rho_A(x) = (x-2)(x+1)$, so A has distinct eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, and thus is diagonalizable.

Solve $(2I - A)\mathrm{x} = 0$: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -3 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

has general solution $\mathrm{x} = \begin{bmatrix} s \\ s \end{bmatrix}$, $s \in \mathbb{R}$, so one possible eigenvector is $\mathrm{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Solve $(-I - A)\mathrm{x} = 0$: $\left[\begin{array}{cc|c} -3 & 0 & 0 \\ -3 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

has general solution $\mathrm{x} = \begin{bmatrix} 0 \\ t \end{bmatrix}$, $t \in \mathbb{R}$, so one possible eigenvector is $\mathrm{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus, set $P := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$,

and $D = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

Example (continued)

Therefore,

$$\begin{aligned}
V_k &= A^k V_0 \\
&= PD^k P^{-1} V_0 \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^k \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 2^k & 0 \\ 2^k & (-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 2^k \\ 2^k - 2(-1)^k \end{bmatrix}
\end{aligned}$$

Remark

Often, instead of finding an exact formula for V_k, it suffices to estimate V_k as k gets large.

This can easily be done if A has a dominant eigenvalue with multiplicity one: an eigenvalue λ_1 with the property that

$$|\lambda_1| > |\lambda_j| \text{ for } j = 2, 3, \ldots, n.$$

Suppose that

$$V_k = PD^k P^{-1} V_0,$$

and assume that A has a dominant eigenvalue, λ_1, with a corresponding eigenvector x_1 as the first column of P.

For convenience, write $P^{-1}V_0 = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$.

Then

$$
\begin{aligned}
V_k &= PD^kP^{-1}V_0 \\
&= \left[\begin{array}{cccc} \mathrm{x}_1 & \mathrm{x}_2 & \cdots & \mathrm{x}_n \end{array}\right] \left[\begin{array}{cccc} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{array}\right] \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}\right] \\
&= b_1\lambda_1^k\mathrm{x}_1 + b_2\lambda_2^k\mathrm{x}_2 + \cdots + b_n\lambda_n^k\mathrm{x}_n \\
&= \lambda_1^k\left(b_1\mathrm{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathrm{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathrm{x}_n\right)
\end{aligned}
$$

Now, $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$ for $j = 2, 3, \ldots n$, and thus $\left(\frac{\lambda_j}{\lambda_1}\right)^k \to 0$ as $k \to \infty$.

Therefore, for large values of k, $\boxed{V_k \approx \lambda_1^k b_1 \mathrm{x}_1}$.

Example

If $A := \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$, and $V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, estimate $V_k = A^k V_0$ for large values of k.

Solution. In our previous example, we found that A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. This means that $\lambda_1 = 2$ is a dominant eigenvalue.

As before $\mathrm{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 2$, and $\mathrm{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -1$, giving us

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and } P^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

$$P^{-1}V_0 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

For large values of k,

$$\boxed{V_k \approx \lambda_1^k b_1 \mathrm{x}_1 = 2^k(1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2^k \\ 2^k \end{bmatrix}}$$

Compare this approximation to the exact formula for V_k that we obtained earlier:

$$V_k = \begin{bmatrix} 2^k \\ 2^k - 2(-1)^k \end{bmatrix}$$

Student Individual Final PROBLEM 1

$$\text{Let} \quad V_{k+1} = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} V_k, \quad \text{and } V_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Let $A := \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$. We will go through the steps to approximate V_k for large k.

a. By hand, compute the characteristic polynomial of A. Use it to compute the eigenvalues of A. One of the eigenvalues is the dominant eigenvalue λ; what is it?

b. Write down D, a diagonal matrix whose entries are the eigenvalues you computed. Put the dominant eigenvalue λ on the first column of D. Write down an invertible P so that $A = PDP^{-1}$. (There are different ways to do this. Please use the algorithm covered in class lectures www.youtube.com/channel/UC2UZ2jPm5y7T2rvLtZY9llg)

c. Compute P^{-1} by hand using the row reduce algorithm, but check with a software afterwards.

d. For large values of k, use the dominant eigenvalue, P, and P^{-1} to estimate $A^k V_0$. (The easiest way is to follow the steps shown in the slides above!)

Student Individual Final PROBLEM 2

$$\text{Let} \quad V_{k+1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix} V_k, \quad \text{and } V_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $B := \left[\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{smallmatrix}\right]$. I have computed that the eigenvalues of B: $5, 1, -2$. (What is the dominant eigenvalue?)

a. Using a software (or by hand), find 3 eigenvectors corresponding to the three eigenvalues. Use them to find an eigenbasis for B. (There are different methods. Use the algorithm covered in class lectures www.youtube.com/channel/UC2UZ2jPm5y7T2rvLtZY9llg)

b. Let $D := \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Use the eigenbasis to write down an invertible matrix P such that $B = PDP^{-1}$.

c. Compute P^{-1} using the row reduce algorithm. Use a software to do the row reduce (or do by hand). Write down your input and output.

d. Use P^{-1}, P and the dominant eigenvalue to approximate V_k for large k, as shown above. (Use the steps shown in the slides above!)