An Application to Systems of Differential Equations

If f and g are differentiable functions, a system

$$\begin{aligned} f' &= 3f + 5g \\ g' &= -f + 2g \end{aligned}$$

is called a *system of differential equations.* Solving many practical problems often comes down to finding sets of functions that satisfy such a system (often involving more than two functions). In this section we show how diagonalization can help. Of course an acquaintance with calculus is required.

The Exponential Function

The simplest differential system is the following single equation:

$$f' = af \text{ where } a \text{ is constant} \tag{0.1}$$

It is easily verified that $f(x) = e^{ax}$ is one solution; in fact, Equation 0.1 is simple enough for us to find *all* solutions.

Theorem 1. *The set of solutions to $f' = af$ is $\{ce^{ax} \mid c \text{ any constant}\}$.*

Remarkably, this result together with diagonalization enables us to solve a wide variety of differential systems.

General Differential Systems

Solving a variety of problems, particularly in science and engineering, comes down to solving a system of linear differential equations. follows. The general problem is to find differentiable functions $f_1, f_2, \dots, f_n$ that satisfy a system of equations of the form

$$\begin{aligned} f_1{}' &= a_{11}f_1 + a_{12}f_2 + \cdots + a_{1n}f_n \\ f_2{}' &= a_{21}f_1 + a_{22}f_2 + \cdots + a_{2n}f_n \\ \vdots \quad & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ f_n{}' &= a_{n1}f_1 + a_{n2}f_2 + \cdots + a_{nn}f_n \end{aligned}$$

where the a_{ij} are constants. This is called a **linear system of differential equations** or simply a **differential system**. The first step is to put it in matrix form. Write

$$f := \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \quad f' := \begin{bmatrix} f_1{}' \\ f_2{}' \\ \vdots \\ f_n{}' \end{bmatrix} \quad A =: \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then the system can be written compactly using matrix multiplication:

$$f' = Af$$

Hence, given the matrix A, the problem is to find a column f of differentiable functions that satisfies $f' = Af$. This can be done if A is diagonalizable. Here is an example.

Theorem 2. *Consider a linear system*

$$f' = Af$$

of differential equations, where A is an $n \times n$ diagonalizable matrix. Let $\{x_1, x_2, \dots, x_n\}$ be an eigenbasis for A with eigenvectors corresponding to eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively. Then every solution f of $f' = Af$ has the form

$$f(x) = c_1x_1e^{\lambda_1 x} + c_2x_2e^{\lambda_2 x} + \cdots + c_nx_ne^{\lambda_n x}$$

where $c_1, c_2, \dots, c_n$ are in $\mathbb{R}$.

The theorem shows that *every* solution to $f' = Af$ is a linear combination

$$f(x) = c_1x_1e^{\lambda_1 x} + c_2x_2e^{\lambda_2 x} + \cdots + c_nx_ne^{\lambda_n x}$$

where the coefficients c_i are arbitrary. Hence this is called the **general solution** to the system of differential equations. In most cases the solution functions $f_i(x)$ are required to satisfy initial conditions, often of the form $f_i(a) = b_i$, where $a, b_1, \dots, b_n$ are prescribed numbers. These conditions determine the constants c_i.

Example 3. Find a solution to the system

$$\begin{aligned} f_1{}' &= f_1 + 3f_2 \\ f_2{}' &= 2f_1 + 2f_2 \end{aligned}$$

that satisfies $f_1(0) = 0,\ f_2(0) = 5$.

Solution:

- This is $f' = Af$, where $f := \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ and $A := \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.
- Compute the characteristic polynomial of A: $\rho_A(x) = (x-4)(x+1)$, so A has eigenvalues 4 and -1. Since A has distinct eigenvalues, it is diagonalizable.
- The vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are eigenvectors corresponding to 4 and 1, respectively. They form an eigenbasis for A.
- By Theorem 2, the *general solution* is

$$\boxed{\begin{aligned} f_1(x) &= ce^{4x} + 3de^{-x} \\ f_2(x) &= ce^{4x} - 2de^{-x} \end{aligned}} \quad c \text{ and } d \text{ constants}$$

  Note this can be written in matrix form as

$$\boxed{\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x} + d \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-x}}$$

- Finally, the requirement that $f_1(0) = 0$ and $f_2(0) = 5$ in this example determines the constants c and d:

$$\begin{aligned} 0 = f_1(0) = ce^0 + 3de^0 = c + 3d \\ 5 = f_2(0) = ce^0 - 2de^0 = c - 2d \end{aligned}$$

  We perform row reduce on the augmented matrix corresponding to the above. We get one unique solution: $c = 3$ and $d = -1$.

  So our final answer is

$$\begin{aligned} f_1(x) = 3e^{4x} - 3e^{-x} \\ f_2(x) = 3e^{4x} + 2e^{-x}. \end{aligned}$$

The following example illustrates this and displays a situation where one eigenvalue has multiplicity greater than 1.

Example 4. Find the general solution to the system

$$\begin{aligned} f_1{}' &= 5f_1 + 8f_2 + 16f_3 \\ f_2{}' &= 4f_1 + f_2 + 8f_3 \\ f_3{}' &= -4f_1 - 4f_2 - 11f_3 \end{aligned}$$

Then find a solution where the initial conditions $f_1(0) = 1$, $f_2(0) = 1$, and $f_3(0) = 1$ are satisfied.

Solution:

- The system has the form $f' = Af$, where $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$.
- Compute the characteristic polynomial:$\rho_A(x) = (x+3)^2(x-1)$.
- An eigenbasis computed (using the method given in Lectures 15) has eigenvectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix},$$

corresponding to the eigenvalues -3, -3, and 1, respectively.

- By Theorem 2, the general solution is

$$f(x) = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-3x} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{-3x} + c_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} e^{x}, \quad c_i \text{ constants.}$$

- The initial conditions $f_1(0) = f_2(0) = f_3(0) = 1$ determine the constants c_1, c_2, c_3.

$$\begin{aligned} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = f(0) = c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \\ = \begin{bmatrix} -1 & -2 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \end{aligned}$$

After performing row reduce on the augmented matrix corresponding to the above, we get a row echelon form with leading 1 in each column, so there is one unique solution: $c_1 = -3$, $c_2 = 5$, $c_3 = 4$.

So the specific solution is

$$\boxed{\begin{aligned} f_1(x) &= -7e^{-3x} + 8e^{x} \\ f_2(x) &= -3e^{-3x} + 4e^{x} \\ f_3(x) &= 5e^{-3x} - 4e^{x} \end{aligned}}$$

1 Student Individual Final PROBLEM 1

Consider the system of differential equations
$$\begin{aligned} f_1{}' &= 2f_1 + 4f_2, \quad f_1(0) = 0. \\ f_2{}' &= 3f_1 + 3f_2, \quad f_2(0) = 1 \end{aligned}$$

a Write A such that $\begin{bmatrix} f_1{}' \\ f_2{}' \end{bmatrix} = A \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

b **By hand**, compute the characteristic polynomial of A. Use the convention given in lecture www.youtube.com/watch?v=4eNHakKpp30 Use it to write down the eigenvalues of A.

c **By hand**, try to compute an eigenbasis of A (if it exists) using the eigenbasis algorithm taught in Math 3333.

d Use Theorem 1 to find the general solution of the system using the eigenvalues and eigenbasis you computed. Then find the specific solution satisfying $f_1(0) = 0, f_2(0) = 1$.

2 Student Individual Final PROBLEM 2

Consider the system of differential equations
$$\begin{aligned} f' &= \phantom{-f + {}} 4g + 4h \\ g' &= f + g - 2h \\ h' &= -f + g + 4h \end{aligned}$$

a Write B such that $\begin{bmatrix} f' \\ g' \\ h' \end{bmatrix} = B \begin{bmatrix} f \\ g \\ h \end{bmatrix}$.

b **By hand**, compute the characteristic polynomial of B. (Use the convention given in lecture www.youtube.com/watch?v=4eNHakKpp30.) Use it to find all eigenvalues of B. (Hint: one of the eigenvalues is 0, so the polynomial is easy to factor.)

c **By hand**, compute all eigenvectors corresponding to the eigenvalue **0**. What is the dimension of the 0-eigenspace?

d Compute an eigenbasis for B using the algorithm taught in Math 3333. (Hint: If your B was computed correctly, an eigenbasis exists.) You can use a calculator to do the row reduce. Show the input you type in and the output the calculator gives you.

e Use Theorem 1 to find the general solution of the system. Then find the specific solution satisfying

$$f(0) = 1, g(0) = 1, h(0) = 1.$$