Lecture 16b

Vector Spaces (subspaces)

Review

Last time: Almost every definition in this class can be defined using addition + scalar multiplication

(Recall) Definition 1: A vector space

A **vector space** is a set V in which

- there is a rule to add any two elements v, w in V, and
- there is a rule to multiply any v in V by any scalar r in $\mathbb{R}$,

such that eight axioms hold.

Idea: a vector space is a set of objects that behave like a set of vectors.
Examples of vector spaces:

- $\mathbb{R}^4$ (the set of vectors of height 4)
- $\mathbb{P}$ (the set of polynomials in x)
- $\mathbb{P}_2$ (the set of polynomials of degree at most 2)
- $\mathbb{S}$ (the set of sequences)
- $\mathbb{R}^{4\times5}$ (the set of 4×5 matrices)
- $\mathcal{C}^\infty$ (the set of smooth functions of x)

A direct proof that a set is a vector space is tedious because it requires $\underbrace{\text{eight}}_{\text{8 is a lot!}}$ proofs (one for each axiom).

There is one situation in which we don't have to check the axioms.

Making vector spaces out of bigger vector spaces

Let's say we already know V is a vector space.
Given a subset W of V, we can define addition and scalar multiplication in W by restricting the existing definitions in V.

- **This only works if the sum of two elements in W is still in W, and the scalar multiple of an element in W is still in W!**
- **If this works, all eight axioms are free, because they hold in V!**

Intuitively, W inherits the nice properties from V.

Example

Let's say we already know $\mathbb{P}$ is a vector space, but not $\mathbb{P}_3$.

- To add two polynomials in $\mathbb{P}_3$, add them as polynomials in $\mathbb{P}$, and observe the result is still in $\mathbb{P}_3$.
- Scalar multiplication is defined the same way, and also does not leave $\mathbb{P}_3$.

Since the axioms hold in $\mathbb{P}$, they automatically hold in $\mathbb{P}_3$. **So $\mathbb{P}_3$ is a vector space.**

A non-example:

Consider $S :=$ { polynomials of degree exactly 3 }.

Then $x^3 + x$ and $-x^3$ are in S,

but their sum is not.

So S is not closed under addition.

We already have a name for this phenomenon; let's reuse it.

Defintion 4: Subspace of a vector space

Let V be a vector space. A **subspace** of V is a non-empty subset W of V which is...

o.

1. • **closed under addition**; that is,

$$\text{for all } \mathrm{v}, \mathrm{w} \text{ in } W, \text{ the sum } \mathrm{v} + \mathrm{w} \text{ is in } W, \text{ and}$$

2. • **closed under scalar multiplication**; that is,

$$\text{for all } \mathrm{v} \text{ in } W \text{ and } c \text{ in } \mathbb{R}, \text{ the product } c\mathrm{v} \text{ is in } W.$$

$\cancel{\mathbb{R}^n}$ vector space

Whenever this happens, we get the axioms for free. That is...

Fact 7 (Subspaces are vector spaces)

A subspace of a vector space is also a vector space.

• Many of the ideas (subspaces, bases, dimensions) from $\mathbb{R}^n$ will generalize to other vector spaces.

Idea: Use Def 4 to either show that a nonempty subset is a subspace (by checking properties 1 and 2) or show that it's not a subspace (by coming up with a counterexample)

Exercise 2

Let V be the set of polynomials with a factor of $(x+1)$; that is,

$$V := \{(x+1)f \mid f \text{ in } \mathbb{P}\}$$

Show that V is a subspace of $\mathbb{P}$.

Exercise 3

Let W be the set of sequences beginning with 1; that is,

$$W := \{(1, x_1, x_2, x_3, \dots) \mid x_i \text{ in } \mathbb{R}\}$$

Show that W is not a subspace of $\underbrace{\mathbb{S}}_{\text{the vector space of all sequences}}$.

Exercise 2 page 1/4

Let V be the set of polynomials with a factor of $(x+1)$; that is,

$$V := \{(x+1)f \mid f \text{ in } \mathbb{P}\}$$

Show that V is a subspace of $\mathbb{P}$.

[We need to show that 0. V is non-empty
1. V is closed under addition
2. V is closed under multiplication.]

0. Since $(x+1)$ is a polynomial with a factor of $(x+1)$, $(x+1)$ is in V. So V is non empty

[Alternatively, I could have checked that the zero element of $\mathbb{P}$ is in V to show that V is non empty.
The zero element of $\mathbb{P}$ is just the zero polynomial, 0.
$0 = (x+1)0$, so the zero polynomial is in V.]

1. $\underbrace{\text{Let } p \text{ and } q \text{ be in } V.}$ This means $p = (x+1)f$ for some polynomial f,
$q = (x+1)g$ for some polynomial g.

Start w/ "Let letter 1 and letter 2 be in the subset"

We write down what it means for p and q to be in V.

Exercise 2 page 2/4

Let V be the set of polynomials with a factor of $(x+1)$; that is,

$$V := \{(x+1)f \mid f \text{ in } \mathbb{P}\}$$

Show that V is a subspace of $\mathbb{P}$.

0. Since $(x+1)$ is a polynomial with a factor of $(x+1)$, $(x+1)$ is in V. So V is nonempty

1. Let p and q be in V. This means $p=(x+1)f$ for some polynomial f,
$q=(x+1)g$ for some polynomial g.

[We need to show that $p+q$ is in V, i.e, $p+q$ is a polynomial with a factor of $(x+1)$]

$$p+q = (x+1)f + (x+1)g$$
$$= (x+1)[f+g]$$

This is $(x+1)$ times (a polynomial)

Therefore, $p+q$ is in V.

So V is closed under addition.

Exercise 2 page 3/4

Let V be the set of polynomials with a factor of $(x+1)$; that is,

$$V := \{(x+1)f \mid f \text{ in } \mathbb{P}\}$$

Show that V is a subspace of $\mathbb{P}$.

0. Since $(x+1)$ is a polynomial with a factor of $(x+1)$, $(x+1)$ is in V. So V is nonempty

1. Let p and q be in V. This means $p = (x+1)f$ for some polynomial f,
$q = (x+1)g$ for some polynomial g.

$$p+q = (x+1)f + (x+1)g$$
$$= (x+1)[f+g]$$

Therefore, $p+q$ is in V. So V is closed under addition.

2. Let c be in $\mathbb{R}$ and p in V. [To show a subset is closed under scalar multiplication, always start with "Let a letter be in $\mathbb{R}$ and let another letter be in the subset"]

That is, $p = (x+1)f$ for some f in $\mathbb{P}$ [Write down what it means for p to be in V]

[We need to check that cp is in V]

Then $cp = c(x+1)f$
$= (x+1)[cf]$ ⇐ Note: cf is a polynomial

Therefore, cp is in V. So V is closed under scalar multiplication.

Thus V is a subspace of $\mathbb{P}$.

—the end—

Exercise 2 page 4/4

Let V be the set of polynomials with a factor of $(x+1)$; that is,

$$V := \{(x+1)f \mid f \text{ in } \mathbb{P}\}$$

Show that V is a subspace of $\mathbb{P}$.

SAMPLE STUDENT ANSWER

0. Since $(x+1)$ is a polynomial with a factor of $(x+1)$,
   $(x+1)$ is in V. So V is nonempty

1. Let p and q be in V. This means $p = (x+1)f$ for some polynomial f,
   $q = (x+1)g$ for some polynomial g.

$$p+q = (x+1)f + (x+1)g$$
$$= (x+1)[f+g]$$

Therefore, $p+q$ is in V. So V is closed under addition.

2. Let c be in $\mathbb{R}$ and p in V.
   That is, $p = (x+1)f$ for some f in $\mathbb{P}$.
   Then $cp = c(x+1)f$
   $= (x+1)[cf]$

Therefore, cp is in V. So V is closed under scalar multiplication.

Thus V is a subspace of $\mathbb{P}$.

— the end —

Exercise 3 pg 1/2

Let W be the set of sequences beginning with 1; that is,

$$W := \{(1, x_1, x_2, x_3, \ldots) \mid x_i \text{ in } \mathbb{R}\}$$

Show that W is not a subspace of $\mathbb{S}$.

We need to show that one of these properties

0. V is non-empty
1. V is closed under addition
2. V is closed under scalar multiplication.

fails by giving one concrete counterexample.

I see that W is non-empty, for example, the sequence $(1,1,1,\ldots)$ is in W,
so I should find either two sequences in W such that their sum is not in W
or one number c and one sequence in W such that c times the sequence is not in W.

Possible answer 1:

Let $a = (1,1,1,\ldots)$ which is in W.

Then $a+a = (2,2,2,\ldots)$

Since the first term of $a+a$ is $2 \neq 1$, the sequence $a+a$ is not in W.

So W is not closed under addition.

Therefore W is not a subspace of $\mathbb{S}$.

Exercise 3 pg 2/2

Let W be the set of sequences beginning with 1; that is,

$$W := \{(1, x_1, x_2, x_3, \ldots) \mid x_i \text{ in } \mathbb{R}\}$$

Show that W is <u>not</u> a subspace of $\mathbb{S}$.

We need to show that one of these properties

0. V is non-empty
1. V is closed under addition
2. V is closed under scalar multiplication.

fails by giving one <u>concrete</u> counterexample.

<u>Possible answer 2:</u>

Let $a := (1,1,1,\ldots)$ which is in W.

Then $5a = (5,5,5,\ldots)$.

Since the first term of $5a$ is $5 \neq 1$, the sequence $5a$ is not in W.

So W is not closed under scalar multiplication.

Therefore W is not a subspace of $\mathbb{S}$.

More examples of vector spaces constructed as subspaces

- Our favorite subspaces of $\mathbb{R}^n$:
  - Images, spans, kernels, eigenspaces, and solutions to HSLEs.
- Each $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.
- The symmetric 3×3 matrices form a subspace of $\mathbb{R}^{3\times 3}$.
- $\mathbb{P}$ is a subspace of $\mathcal{C}^\infty$.

$\mathbb{P}_n$ is a subspace of $\mathbb{P}$,
$\mathbb{P}$ is a subspace of C^∞.

Recall:

- $\mathcal{C}^\infty$ denotes the set of smooth functions in x
- $\mathcal{C}^\infty$ is a vector space

Exercise 4

Let S denote the set of smooth functions $f(x)$ in $\mathcal{C}^\infty$ such that $f''(x) = -f(x)$. That is,

$$S = \{f(x) \text{ in } \mathcal{C}^\infty \mid f''(x) = -f(x)\}.$$

Show that S is a subspace of $\mathcal{C}^\infty$.

According to Definition 4, we need to show that ...

0. we can name a smooth function f in C^∞ where $f'' = -f$
1. S is closed under addition
2. S is closed under scalar multiplication.

- $\mathcal{C}^\infty$ denotes the set of smooth functions in x
- $\mathcal{C}^\infty$ is a vector space

Exercise 4 pg 1/4

Let S denote the set of smooth functions $f(x)$ in $\mathcal{C}^\infty$ such that $f''(x) = -f(x)$. That is,

$$S = \{f(x) \text{ in } \mathcal{C}^\infty \mid f''(x) = -f(x)\}.$$

Show that S is a subspace of $\mathcal{C}^\infty$.

S is the set of solutions to the differential equation $f''(x) = -f(x)$.

0. Come up with just one solution to $f'' = -f$.

- Recall $\sin(x)$ is smooth (all higher derivatives of $\sin(x)$ exists)

$$\frac{d}{dx}\sin(x) = \cos(x), \quad \frac{d}{dx}\cos(x) = -\sin(x),$$

so $$\frac{d^2}{dx^2}\sin(x) = \frac{d}{dx}\cos(x) = -\sin(x)$$

so $\sin(x)$ is in S

- The zero function also works: $\frac{d^2}{dx^2}0 = 0 = -0$

0. The function $\sin(x)$ is smooth (all derivatives of $\sin(x)$ exist),

and $$\frac{d^2}{dx^2}\sin(x) = \frac{d}{dx}\cos(x) = -\sin(x).$$

So S is non-empty.

- $\mathcal{C}^\infty$ denotes the set of smooth functions in x
- $\mathcal{C}^\infty$ is a vector space

Exercise 4 Pg 2/4

Let S denote the set of smooth functions $f(x)$ in $\mathcal{C}^\infty$ such that $f''(x) = -f(x)$. That is,

$$S = \{f(x) \text{ in } \mathcal{C}^\infty \mid f''(x) = -f(x)\}.$$

Show that S is a subspace of $\mathcal{C}^\infty$.

S is the set of solutions to the differential equation $f''(x) = -f$.

0. The function $\sin(x)$ is smooth (all derivatives of $\sin(x)$ exists), and $\frac{d^2}{dx^2}\sin(x) = \frac{d}{dx}\cos(x) = -\sin(x)$. So S is non-empty.

[1. To show S is closed under addition, write "let function 1 and function 2 be in S".
Write down what it means for function 1 and function 2 to be in S.
Do computation which shows function 1 + function 2 is also in S.]

1. Let f and g be in S. That is, f and g are smooth and

$$f'' = -f$$
$$g'' = -g.$$

Then $f+g$ is also smooth (since $\mathcal{C}^\infty$ is a vector space, $\mathcal{C}^\infty$ is closed under addition), and

$$\begin{aligned}\frac{d^2}{dx^2}(f+g) &= \left(\frac{d^2}{dx^2}f\right) + \left(\frac{d^2}{dx^2}g\right)\\ &= f'' + g''\\ &= -f + -g\\ &= -(f+g).\end{aligned}$$

Therefore, $f+g$ is in S, so S is closed under addition.

- $\mathcal{C}^\infty$ denotes the set of smooth functions in x
- $\mathcal{C}^\infty$ is a vector space

Exercise 4 pg 3/4

Let S denote the set of smooth functions $f(x)$ in $\mathcal{C}^\infty$ such that $f''(x) = -f(x)$. That is,

$$S = \{f(x) \text{ in } \mathcal{C}^\infty \mid f''(x) = -f(x)\}.$$

Show that S is a subspace of $\mathcal{C}^\infty$.

S is the set of solutions to the differential equation $f''(x) = -f$.

2. To show S is closed under scalar multiplication, write
"Let a letter c be in $\mathbb{R}$ and let another letter f be in S".
Write down what it means for another letter f to be in S.
Do computation showing cf is still in S

2. Let c be in $\mathbb{R}$ and let f be in S.
That is, f is smooth and $f'' = -f$.
Then cf is smooth (because C^∞ is a vector space, so C^∞ is closed under scalar multiplication)

$$\begin{aligned}(cf)'' &= c f'' \\ &= c(-f) \\ &= -(cf).\end{aligned}$$

So cf is in S. Therefore S is closed under scalar multiplication.
Hence S is a subspace of C^∞.

— the end —

- $\mathcal{C}^\infty$ denotes the set of smooth functions in x
- $\mathcal{C}^\infty$ is a vector space

Exercise 4 Pg 4/4

Let S denote the set of smooth functions $f(x)$ in $\mathcal{C}^\infty$ such that $f''(x) = -f(x)$. That is,

$$S = \{f(x) \text{ in } \mathcal{C}^\infty \mid f''(x) = -f(x)\}.$$

Show that S is a subspace of $\mathcal{C}^\infty$.

SAMPLE STUDENT ANSWER

0. The function $\sin(x)$ is smooth (all derivatives of $\sin(x)$ exists), and $\frac{d^2}{dx^2}\sin(x) = \frac{d}{dx}\cos(x) = -\sin(x)$. So S is non-empty.

1. Let f and g be in S. That is, f and g are smooth and

$$f'' = -f$$
$$g'' = -g.$$

Then $f+g$ is also smooth (since $\mathcal{C}^\infty$ is a vector space, $\mathcal{C}^\infty$ is closed under addition), and

$$\begin{aligned}\frac{d^2}{dx^2}(f+g) &= \left(\frac{d^2}{dx^2}f\right) + \left(\frac{d^2}{dx^2}g\right)\\ &= f'' + g''\\ &= -f + -g\\ &= -(f+g).\end{aligned}$$

Therefore, $f+g$ is in S, so S is closed under addition.

2. Let c be in $\mathbb{R}$ and let f be in S. That is, f is smooth and $f'' = -f$.

Then cf is smooth (because $\mathcal{C}^\infty$ is a vector space, so $\mathcal{C}^\infty$ is closed under scalar multiplication)

$$\begin{aligned}(cf)'' &= c f''\\ &= c(-f)\\ &= -(cf).\end{aligned}$$

So cf is in S. Therefore S is closed under scalar multiplication.

Hence S is a subspace of $\mathcal{C}^\infty$.

— the end —

In the last exercise, we saw an example of the following theorem.

Theorem 8 (Differential equations and linear algebra)

The solutions to a linear differential equation form subspace of $\mathcal{C}^\infty$

This allows us to use the techniques of linear algebra to study the vector space of solutions to a given linear differential equation. For example ...

Corollary

If $f_1, f_2, ..., f_n$ are solutions to a linear differential equation, then any linear combination

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

is also a solution.

Recall:

- $\mathbb{P}_2$ denotes the set of polynomials $f(x)$ of degree at most 2.
- $\mathbb{P}_2$ is a vector space.

Exercise 5(a)

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

Show whether S is a subspace or not a subspace of $\mathbb{P}_2$.

Exercise 5(b)

Let T denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 1$. That is,

$$T = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 1\}.$$

Show whether T is a subspace or not a subspace of $\mathbb{P}_2$.

(Here, $f(5)$ means 'plug in 5 for x'.)

- $\mathbb{P}_2$ denotes the set of polynomials $f(x)$ of degree at most 2.
- $\mathbb{P}_2$ is a vector space.

Exercise 5(a) pg 1/3

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

Show whether S is a subspace or not a subspace of $\mathbb{P}_2$.

[Try to show that S is a subspace:

0. S is non-empty; 1. S is closed under addition; 2. S is closed under scalar multiplication.]

0. The polynomial $\underbrace{x-5}_{\text{degree is } 1 \leq 2}$ is in $\mathbb{P}_2$ and plugging in 5 into $x-5$ gives 0.

   So $x-5$ is in S. This shows S is non empty.

1. Let f and g be in S.
   That is, f and g are polynomials with degree 2 or smaller, and

   $$f(5)=0$$
   $$g(5)=0.$$

   So $f+g$ is a polynomial with degree 2 or smaller, and

   $$\begin{aligned}(f+g)(5) &= f(5)+g(5)\\ &= 0+0\\ &= 0\end{aligned}$$

   Therefore $f+g$ is in S. So S is closed under addition.

Looks promising!

- $\mathbb{P}_2$ denotes the set of polynomials $f(x)$ of degree at most 2.
- $\mathbb{P}_2$ is a vector space.

Exercise 5(a) Pg 2/3

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

Show whether S is a subspace or not a subspace of $\mathbb{P}_2$.

[Try to show that S is a subspace:
0. S is non-empty; 1. S is closed under addition;
2. S is closed under scalar multiplication.]

2. Let c be in $\mathbb{R}$ and let f be in S.
That is, f is a polynomial in x with degree 2 or smaller, and

$f(5) = 0$.

Then cf is also in $\mathbb{P}_2$ and

$$\begin{aligned}(cf)(5) &= c \cdot f(5) \\ &= c \cdot 0 \\ &= 0\end{aligned}$$

So cf is in S. Therefore S is closed under scalar multiplication.

Thus, S is a subspace of $\mathbb{P}_2$.

— the end —

- $\mathbb{P}_2$ denotes the set of polynomials $f(x)$ of degree at most 2.
- $\mathbb{P}_2$ is a vector space.

Exercise 5(a) pg 3/3

Let S denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 0$. That is,

$$S = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 0\}.$$

Show whether S is a subspace or not a subspace of $\mathbb{P}_2$.

SAMPLE STUDENT ANSWER

0. The polynomial $\underbrace{x-5}_{\text{degree is } 1 \leq 2}$ is in $\mathbb{P}_2$ and plugging in 5 into $x-5$ gives 0.
So $x-5$ is in S. This shows S is nonempty.

1. Let f and g be in S.
That is, f and g are polynomials with degree 2 or smaller, and

$$f(5)=0$$
$$g(5)=0.$$

So $f+g$ is a polynomial with degree 2 or smaller, and

$$(f+g)(5) = f(5)+g(5)$$
$$= 0+0$$
$$= 0$$

Therefore $f+g$ is in S. So S is closed under addition.

2. Let c be in $\mathbb{R}$ and let f be in S.
That is, f is a polynomial in x with degree 2 or smaller, and

$$f(5) = 0.$$

Then cf is also in $\mathbb{P}_2$ and

$$(cf)(5) = c.f(5)$$
$$= c.0$$
$$= 0$$

So cf is in S. Therefore S is closed under scalar multiplication.

Thus, S is a subspace of $\mathbb{P}_2$.

— the end —

Exercise 5(b) pg 1/2

Let T denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 1$. That is,

$$T = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 1\}.$$

Show whether T is a subspace or not a subspace of $\mathbb{P}_2$.

My Scratch work (Don't submit your scratch work!)

Try to show T is a subspace: 0. T is nonempty
1. T is closed under addition
2. T is closed under scalar multiplication

$x-4$ is in $\mathbb{P}_2$ and plugging in 5 into $x-4$ gives 1, so T is nonempty.

But if I have two polynomials f, g in T, then $(f+g)(5) = f(5) + g(5) = 1+1 = 2$.

This is not a concrete counterexample!!

Let $f(x) := x-4$, which is in T.

Then $(f+f)(x) = x-4 + x-4$
$= 2x-8$

so $(f+f)(5) = 2(5)-8$
$= 2$

Since $(f+f)(5) \neq 1$, $f+f$ is not in T.

So T is not closed under addition. Therefore T is not a subspace of $\mathbb{P}_2$.

Exercise 5(b) pg 2/2

Let T denote the set of polynomials in $\mathbb{P}_2$ such that $f(5) = 1$. That is,

$$T = \{f(x) \text{ in } \mathbb{P}_2 \mid f(5) = 1\}.$$

Show whether T is a subspace or not a subspace of $\mathbb{P}_2$.

SAMPLE STUDENT ANSWER

Let $f(x) := x-4$, which is in T.

Then $(f+f)(x) = x-4 + x-4$
$= 2x-8$

so $(f+f)(5) = 2(5)-8$
$= 2$

Since $(f+f)(5) \neq 1$, $f+f$ is not in T.
So T is not closed under addition.

Therefore T is not a subspace of $\mathbb{P}_2$.

— the end —

ANOTHER SAMPLE STUDENT ANSWER

Let $f(x) := x-4$, which is in T.

Then $(4f)(x) = 4(x-4)$
$= 4x-16$.

So $(4f)(5) = 20-16$
$= 4$

Since $(4f)(5) \neq 1$,

$4f$ is not in T.

So T is not closed under scalar multiplication.

Therefore T is not a subspace of $\mathbb{P}_2$.

— the end —