Lecture 15b

Eigenbases (finding eigenbases)

Review

Recall: Definition (Eigenbases)

An **eigenbasis** for an $n \times n$-matrix A is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

1

2

Recall (Exercise 1 from the last lecture)

The following vectors form an eigenbasis for A.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\} \qquad A := \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

Observation 1 (Matrix multiplication and eigenbases)

Let $v_1, v_2, \dots, v_n$ be an eigenbasis for A, and let λ_i denote the eigenvalue of v_i.
If $w = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$, then

Use eigenbasis to speed up multiplication!

$$Aw = c_1\lambda_1 v_1 + c_2\lambda_2 v_2 + \cdots + c_n\lambda_n v_n$$
$$A^2 w = c_1\lambda_1^2 v_1 + c_2\lambda_2^2 v_2 + \cdots + c_n\lambda_n^2 v_n$$
$$A^m w = c_1\lambda_1^m v_1 + c_2\lambda_2^m v_2 + \cdots + c_n\lambda_n^m v_n$$

Algorithm 2 (How to find an eigenbasis)

We are given an $n \times n$ matrix A.

1 • Find the eigenvalues of A (by factoring the characteristic polynomial.)

2 • For each eigenvalue, find a basis of the λ-eigenspace.
  - That is, a basis for $\ker(A - \lambda \mathrm{Id})$

3 • Put all the vectors together into a set.
  - If there are n-many vectors, the set is an eigenbasis!
  - If there are fewer than n-many vectors, no eigenbasis exists!

Fact: This algorithm constructs a linearly independent set.

Exercise 4

Find an eigenbasis for $\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ or state it doesn't exist.

(Use Algorithm 2)

Solution to Exercise 4 (page 1/3) using Algorithm 2

Set $A := \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$.

Step ① Find the eigenvalues of A by factoring the characteristic polynomial.

- Find the characteristic polynomial of A:

$$
\begin{aligned}
P_A(x) &= \det(x\mathrm{Id} - A) \\
&= \det\left(\begin{bmatrix} x-1 & -1 \\ -1 & x-3 \end{bmatrix}\right) \\
&= (x-1)(x-3) - 1 \\
&= x^2 - 4x + 3 - 1
\end{aligned}
$$

So $P_A(x) = x^2 - 4x + 2$

- Find roots of $P_A(x)$ (quadratic formula or "complete the square")

The roots are $\dfrac{4 \pm \sqrt{16-8}}{2} = \dfrac{4 \pm \sqrt{8}}{2} = \dfrac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$

So the eigenvalues of A are $\lambda_1 = 2 + \sqrt{2}$ and $\lambda_2 = 2 - \sqrt{2}$

Step ② (i): · Find a basis for the λ_1-eigenspace of A $(\lambda_1 = 2+\sqrt{2})$, that is, find a basis for $\ker(A-(2+\sqrt{2})\,\mathrm{Id})$.

$$A-(2+\sqrt{2})\,\mathrm{Id} = \begin{bmatrix}1&1\\1&3\end{bmatrix} - \begin{bmatrix}2+\sqrt{2}&0\\0&2+\sqrt{2}\end{bmatrix} = \begin{bmatrix}1-2-\sqrt{2}&1\\1&3-2-\sqrt{2}\end{bmatrix} = \begin{bmatrix}-1-\sqrt{2}&1\\1&1-\sqrt{2}\end{bmatrix}$$

Solve for $\left(A-(2+\sqrt{2})\,\mathrm{Id}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$:

$$\left[\begin{array}{cc|c}-1-\sqrt{2}&1&0\\1&1-\sqrt{2}&0\end{array}\right] \xrightarrow[\text{swap } R_1,R_2]{} \left[\begin{array}{cc|c}1&1-\sqrt{2}&0\\-1-\sqrt{2}&1&0\end{array}\right] \xrightarrow[R_2\mapsto(1+\sqrt{2})R_1+R_2]{} \left[\begin{array}{cc|c}1&1-\sqrt{2}&0\\0&\underbrace{(1+\sqrt{2})(1-\sqrt{2})+1}&0\end{array}\right] \qquad \left[\begin{array}{cc|c}1&1-\sqrt{2}&0\\0&0&0\end{array}\right]$$

$$(1+\sqrt{2})(1-\sqrt{2})+1 = 1-(\sqrt{2})^2+1 = 1-2+1 = 0$$

Since the 2nd column has no leading 1, let $y = t$.

Back substitution: $x+(1-\sqrt{2})y = 0 \Rightarrow x+(1-\sqrt{2})t = 0 \Rightarrow x = (-1+\sqrt{2})t$

General solution to $\left(A-(2+\sqrt{2})\,\mathrm{Id}\right)\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ is $\begin{bmatrix}(-1+\sqrt{2})t\\t\end{bmatrix} = t\begin{bmatrix}-1+\sqrt{2}\\1\end{bmatrix}$

A basis for the λ_1-eigenspace of A is $\left\{\begin{bmatrix}-1+\sqrt{2}\\1\end{bmatrix}\right\}$

Solution to Exercise 4 (page 3/3)

Step 2 (ii): Find a basis for the λ_2-eigenspace of A ($\lambda_2 = 2-\sqrt{2}$), that is, find a basis for $\ker(A-(2-\sqrt{2})\,\mathrm{Id})$.

$$A-(2-\sqrt{2})\,\mathrm{Id} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2-\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1-2+\sqrt{2} & 1 \\ 1 & 3-2+\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{bmatrix}$$

Solve for $\left(A-(2-\sqrt{2})\,\mathrm{Id}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$\left[\begin{array}{cc|c} -1+\sqrt{2} & 1 & 0 \\ 1 & 1+\sqrt{2} & 0 \end{array}\right] \xrightarrow[\text{swap } R_1, R_2]{} \left[\begin{array}{cc|c} 1 & 1+\sqrt{2} & 0 \\ -1+\sqrt{2} & 1 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto (1-\sqrt{2})R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & 1+\sqrt{2} & 0 \\ 0 & \underbrace{(1-\sqrt{2})(1+\sqrt{2})+1} & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1+\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$(1-\sqrt{2})(1+\sqrt{2})+1 = 1-(\sqrt{2})^2+1 = 1-2+1 = 0$

Since the 2nd column has no leading 1, let $y=t$.

Back substitution: $x + (1+\sqrt{2})y = 0 \Rightarrow x + (1+\sqrt{2})t = 0 \Rightarrow x = (-1-\sqrt{2})t$

General solution to $\left(A-(2-\sqrt{2})\,\mathrm{Id}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} (-1-\sqrt{2})\,t \\ t \end{bmatrix} = t\begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix}$

A basis for the λ_2-eigenspace of A is $\left\{ \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix} \right\}$

Step 3 Put all vectors together into a set. If there are n vectors, it's an eigenbasis! (otherwise, none exists)

The set $\left\{ \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix} \right\}$ is an eigenbasis for A.

Exercise 5

Find an eigenbasis for

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

or state it doesn't exist.

Use Algorithm 2 (How to find an eigenbasis)

Exercise 5

Find an eigenbasis for

$$A := \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

or state it doesn't exist.

Step ①: (Find eigenvalues of A)

- Find characteristic polynomial and compute roots

$$p_A(x) = \det(x\,\mathrm{Id} - A) = \det\left(\begin{bmatrix} x-1 & 1 \\ -1 & x+1 \end{bmatrix}\right) = (x-1)(x+1) + 1 = x^2 - 1 + 1 = x^2$$

$$p_A(x) = x^2$$

- The only root of $p_A(x) = x^2$ is 0.

The only eigenvalue of A is 0.

Step ②: (Find a basis for each eigenspace of A)

- Find a basis for the 0-eigenspace of A, that is, find a basis for $\ker(\underbrace{A - 0\,\mathrm{Id}}_{\text{equal to } A}) = \ker(A)$

Find solutions to $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow[R_2 \mapsto -R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Since 2nd column has no leading 1, let $\boxed{y = t}$

$x - y = 0 \Rightarrow x - t = 0 \Rightarrow \boxed{x = t}$

General solution is $\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

A basis for the 0-eigenspace of A is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Step ③: Since $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ has fewer than $n = 2$ vectors, no eigenbasis exists.

— the end —

Fact 3 (When are there enough eigenvectors to have an eigenbasis?)

A matrix A has an eigenbasis if and only if

shorthand for the λ-eigenspace of A

$$\mathrm{width}(A) = \sum_{\text{eigenvalues } \lambda} \dim(\underbrace{E_\lambda(A)})$$

Reasoning: A basis for $E_\lambda(A)$ (the λ-eigenspace of A) has $\dim(E_\lambda(A))$-many vectors.

For example, if A is 4×4 with eigenvalues λ_1, λ_2 and the dimension of the λ_1-eigenspace of A is 1 and the dimension of the λ_2-eigenspace of A is 1, then $\mathrm{width}(A) \neq 1+1$ so A has no eigenbasis.

Exercise 6

$$\text{Let } A := \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

Show that A has an eigenbasis (without finding an eigenbasis).
Note: A has two eigenvalues, 1 and 7.

Hint: Compute just $\dim(E_1(A))$ and $\dim(E_7(A))$ then use Fact 3

Exercise 6

Let
$$A := \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

Show that A has an eigenbasis (without finding an eigenbasis).
Note: A has two eigenvalues, 1 and 7.

Find ~~basis~~/ $\underbrace{\text{dimension}}_{\text{faster (we'll compute just the dimension)}}$ of $\underbrace{E_1(A)}_{\text{shorthand for the 1-eigenspace of } A} = \ker(A - 1\,\mathrm{Id})$

$$A - 1\,\mathrm{Id} = \begin{bmatrix} 2-1 & 2 & 1 \\ 2 & 5-1 & 2 \\ 1 & 2 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Since all columns are multiple of each other,
$\operatorname{rank}(A - 1\,\mathrm{Id}) = 1$.

[Recall "Rank-Nullity Theorem":
$\operatorname{width}(M) = \operatorname{rank}(M) + \dim(\ker(M))$]

$$\begin{aligned} \text{So } \dim(\ker(A - 1\,\mathrm{Id})) &= 3 - \operatorname{rank}(A - 1\,\mathrm{Id}) \\ &= 3 - 1 \\ &= 2. \end{aligned}$$

$\therefore$ Dimension of $E_1(A)$ is 2.

Find ~~basis~~/ $\underbrace{\text{dimension}}_{\text{faster to compute}}$ of $\underbrace{E_7(A)}_{\text{the 7-eigenspace of } A}$

Since 7 is an eigenvalue of A,
a 7-eigenvector exists.
So the 7-eigenspace contains a non-zero vector.

Recall: The only subspace with dimension 0 is
the zero subspace. All other subspaces
have dimension 1 or higher.

So $\dim(E_7(A)) \geq 1$.

$$\sum_{\text{eigenvalues } \lambda} \dim(E_\lambda(A)) = \underbrace{\dim(E_1(A))}_{2} + \underbrace{\dim(E_7(A))}_{\text{at least } 1} \geq 3$$

By Fact 3,
$\therefore$ A has an eigenbasis.

A useful trick

If λ is an eigenvalue of A, then

- there is at least one λ-eigenvector, so
- the λ-eigenspace has dimension at least 1, so
- a basis for the λ-eigenspace has at least 1 vector.

Hence, each distinct root of the characteristic polynomial guarantees at least one vector in our potential eigenbasis.

Theorem 4 (*n*-many distinct eigenvalues)

If an $n \times n$-matrix has n-many distinct eigenvalues, it must have an eigenbasis.

Note: If an $n \times n$-matrix has fewer than n distinct eigenvalues, it may have an eigenbasis (We need to use Algorithm 2 or another method to find out for sure).

Exercise 7

Determine whether the following matrix has an eigenbasis.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(Hint: Compute the eigenvalues then use Theorem 4)

Exercise 7

Determine whether the following matrix has an eigenbasis.

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

- Find characteristic polynomial

$$p_A(x) = \det(x\,\mathrm{Id} - A)$$
$$= \det\left(\begin{bmatrix} x-1 & 2 & 3 \\ 0 & x & 0 \\ 1 & 1 & x-1 \end{bmatrix}\right)$$
$$= x\,(-1)^{2+2}\det\left(\begin{bmatrix} x-1 & 3 \\ 1 & x-1 \end{bmatrix}\right)$$
$$= x\left[(x-1)(x-1)-3\right]$$
$$= x\left[x^2-2x+1-3\right]$$
$$p_A(x) = x^3 - 2x^2 - 2x$$

- compute roots

$$p_A(x) = \underbrace{x}_{\lambda_1=0}(x^2-2x-2)$$

One of the roots is $\lambda_1 = 0$

The other two roots ...

$$\frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$\lambda_2 = 1+\sqrt{3}$ and $\lambda_3 = 1-\sqrt{3}$

$\therefore$ By Theorem 4, since A is 3×3 and has 3 distinct eigenvalues, A must have an eigenbasis.

Theorem (A bound from the characteristic polynomial)

Let λ be an eigenvalue of A. Then

$$1 \leq \dim(E_\lambda(A)) \leq (\#\text{ of times } (x-\lambda) \text{ appears in } \rho_A(x))$$

That is, if λ is a root of the characteristic polynomial $\rho_A(x)$ with multiplicity m, then the dimension of the eigenspace is between 1 and m.

Two notions of multiplicity for eigenvalues

There are two different 'ways to count' an eigenvalue.

- $\dim(E_\lambda(A))$ is the **geometric multiplicity** of the eigenvalue λ.
- ($\#$ of times $(x-\lambda)$ appears in $\rho_A(x)$) is the **algebraic multiplicity** of the eigenvalue λ.