Lecture 11a

Subspaces

Recall: Shapes of solutions

The solution set to a system of linear equations (SLE) must be one of the following.

- An empty set no solution
- A point one unique solution
- A line need one parameter, dimension is 1
- A plane need two parameters, dimension is 2
- ...a shape in higher dimension?

This is a remarkably powerful classification!

- The shape is determined by its dimension,
- The shapes all share many nice properties (e.g. contain the line through any pair of points).

We will focus on the "shapes of solutions" for homogeneous system of linear equations (SLE).

Recall (Definition): A system of linear equations (SLE) is called **homogeneous** if the constants are zero; equivalently, the matrix equation is $Av = \vec{0}$.

Shapes of solutions (homogeneous)

The solution set to a homogeneous SLE must be one of:

- The origin
- A line through the origin requires one parameter, dimension: 1
- A plane through the origin requires two parameters, dimension: 2
- ...a higher dimensional shape through the origin?

Why focus on homogeneous systems?

- The properties are nicer.
- The solution set to a general SLE turns out to be a translation of the solution set to a homogeneous SLE.

Goal

Define and study these general planar shapes through the origin.

We cannot use pictures or our geometric intuition for vectors of height 4 or taller.

Our strategy

Find a few essential properties that characterize these planar shapes, and then use those properties as a general definition.

That is, what properties distinguish the solution sets to homogeneous SLEs from all other shapes?

First, we introduce new terminology.

Definition 1: Subsets

Suppose A is a set and B is another set whose elements are all elements of A. Then we say that B is a subset of A. We also say that A contains B.

Examples

- The set $\{1, 3\}$ is a subset of $\{1, 2, 3\}$.
- The set $\{1, 3\}$ is also a subset of the set of real numbers $\mathbb{R}$.
- Any set V consisting some vectors of height 2 is subset of $\mathbb{R}^2$.
- The set $\mathbb{R}^2$ (all vectors of height 2) is a subset of $\mathbb{R}^2$.
- Every set is a subset of itself.
- The set $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$, because a vector of height 2 is not a vector of height 3.
- The set $\{1, \frac{2}{5}\}$ is not a subset of $\mathbb{Z}$ (the set of integers).

Definition 2: "closed under addition" and "closed under scalar multiplication" for subsets

Suppose S is a subset of $\mathbb{R}^n$.

1. We say that S is closed under addition if ...

$$\text{for all } \mathrm{v}, \mathrm{w} \text{ in } S, \text{ the sum } \mathrm{v} + \mathrm{w} \text{ is in } S$$

2. We say that S is closed under scalar multiplication if ...

$$\text{for all v in } S \text{ and } \underbrace{c \text{ in } \mathbb{R}}_{\text{a number } c}, \text{ the product } c\mathrm{v} \text{ is in } S.$$

Exercise 1

Let S be a non-empty subset of $\mathbb{R}^n$ which is closed under scalar multiplication. Show that S must contain the zero vector.

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Solution:

Since S is non-empty and S is a subset of $\mathbb{R}^n$, we know S contains at least one vector of height n.

Let v be a vector in S.

Since S is closed under scalar multiplication, S contains cv for each c in $\mathbb{R}$.

(by definition of "closed under scalar multiplication") (S contains every scalar multiple of v).

Therefore, $0v = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{n\times 1}$ is in S.

— the end —

From now on, we can use this fact without needing to repeat the argument.

Exercise 1

Let S be a non-empty subset of $\mathbb{R}^n$ which is closed under scalar multiplication. Show that S must contain the zero vector.

Whenever we know that a subset of $\mathbb{R}^n$ is nonempty and is closed under scalar multiplication, we also know that this subset contains the zero vector.

Theorem 1 (Properties of solution sets to homogeneous SLEs)

Let V be the solution set to a homogeneous system of linear equations in n variables. Then...

- V is a subset of $\mathbb{R}^n$; that is, the elements of V are n-vectors.
- V contains the zero vector.
- V is closed under addition; that is,

$$\text{for all } \mathrm{v}, \mathrm{w} \text{ in } V, \text{ the sum } \mathrm{v} + \mathrm{w} \text{ is in } V, \text{ and}$$

- V is closed under scalar multiplication; that is,

$$\text{for all } \mathrm{v} \text{ in } V \text{ and } c \text{ in } \mathbb{R}, \text{ the product } c\mathrm{v} \text{ is in } V.$$

There's some overlap between these properties. E.g., Exercise 1 tells us if a subset S is closed under scalar multiplication then S contains the zero vector.

Definition 3: A subspace of $\mathbb{R}^n$

A **subspace** of $\mathbb{R}^n$ is a non-empty subset V of $\mathbb{R}^n$ which is

1. closed under addition; that is,

   for all v, w in V, the sum $\mathrm{v} + \mathrm{w}$ is in V, and

2. closed under scalar multiplication; that is,

   for all v in V and c in $\mathbb{R}$, the product $c\mathrm{v}$ is in V.

Note 1: Exercise 1 tells us a subspace must contain the zero vector.

Examples of subspaces

- The solution set to a homogeneous SLE (by Theorem 1).
- The set of 2-vectors whose entries sum to 0. $\left\{ \begin{bmatrix} t \\ -t \end{bmatrix} \text{ for } t \text{ in } \mathbb{R} \right\}$ line $y = -x$
- The origin.
- A line through the origin.
- A plane through the origin.

Note: By Thm 1, a homogeneous solution set is a subspace.

Definition 4: The kernel of a matrix

Let A be an $m \times n$-matrix. Then the **kernel** of A is the set of vectors v such that $\mathrm{A}\mathrm{v} = \vec{0}$. That is,

$$\ker(\mathrm{A}) := \{\mathrm{v} \text{ in } \mathbb{R}^n \text{ such that } \mathrm{A}\mathrm{v} = \vec{0}\}$$

The textbook calls this set the null space of A, denoted $\mathrm{null}(\mathrm{A})$.

This is just a homogeneous solution set with a different name!

The kernel of A is the same as the solution set to $\mathrm{A}v = \vec{0}$.

Since $\mathrm{A}v = \vec{0}$ is a homogeneous SLE, the kernel of A is the solution set to a homogeneous SLE. So we see from Theorem 1 that...

Theorem 2: Kernels are subspaces

The kernel of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

Exercise 2

$$A := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Determine whether each of the following are in $\ker(A)$.

a) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ d) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ e) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Def 4 says: v is in $\ker(A)$ if $Av = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$ and v in $\mathbb{R}^2$

a), b) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ are not in $\ker(A)$ because $\ker(A)$ is a subset of $\mathbb{R}^2$ (since A has 2 columns)

c) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2+2 \\ -2-2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is not in $\ker(A)$.

d) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2-2 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is in $\ker(A)$. :)

e) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in $\ker(A)$.

Alternative answer to (e):

We've seen that $\boxed{\ker(A) \text{ is a subspace of } \mathbb{R}^2.}$

Thm 2 says: The kernel of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$

(By definition) every subspace of $\mathbb{R}^2$ contains $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is in $\ker(A)$.

— the end of alternative solution to (e) —

Definition 5: The λ-eigenspace of a matrix "lambda"

Let A be an $n \times n$-matrix (square matrix) and λ be a number. The **λ-eigenspace** of A, denoted by $E_\lambda(A)$, is the set of λ-eigenvectors and the zero vector. That is,

$$E_\lambda(A) := \{v \text{ in } \mathbb{R}^n \text{ such that } Av = \lambda v\}$$

Note: $\vec{0}$ is in $E_\lambda(A)$ even though $\vec{0}$ is not an eigenvector.

An eigenspace is just a homogeneous solution set with a different name!

The following three sets are the same.

1. The λ-eigenspace of A.
2. The kernel of the matrix $A - \lambda \mathrm{Id}$.
3. The solution set of the equation $Av = \lambda v$.

Recall: λ-eigenvectors are non-zero solutions to $(A - \lambda \mathrm{Id})v = \vec{0}$

Since the λ-eigenspace of a square matrix is the kernel of a matrix, it follows from Theorem 2 that ...

Theorem 3: Eigenspaces are subspaces

The λ-eigenspace of an $n \times n$ matrix is a subspace of $\mathbb{R}^n$.

So far, all of our subspaces are new perspectives on the same construction: solutions to homogeneous SLEs. Let's give a different source of subspaces.

Definition 6: The image of a matrix

Let A be an $m \times n$-matrix. The **image** of A is the set of vectors v which can be written as v $=$ Aw for some vector w. I.e.

$$\mathrm{im}(\mathsf{A}) := \{\mathsf{v} \text{ in } \mathbb{R}^m \text{ such that } \mathsf{v} = \mathsf{Aw} \text{ for some } \mathsf{w} \text{ in } \mathbb{R}^n\}$$

Theorem 4: Images are subspaces

The image of an $m \times n$ matrix must be a subspace of $\mathbb{R}^m$.

Exercise 3

$$A := \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

A has size 3x2
A has 3 rows
2 columns

Determine whether each of the following are in $\mathrm{im}(A)$.

a) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ e) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\mathrm{im}(A) := \{$v in $\mathbb{R}^m$ such that v $=$ Aw for some w in $\mathbb{R}^n\}$

$= \{$ v in $\mathbb{R}^3$ where $v = Aw$ for some w in $\mathbb{R}^2 \}$

a), b) $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are not in Im(A).

Again, the def says: v is in im(A) if we can find w in $\mathbb{R}^2$ where $Aw = v$.

Strategy to do part (c), (d), (e):

For each vector v, check whether $A \begin{bmatrix} x \\ y \end{bmatrix} = v$ has a solution.

c) Does $\overset{A}{\begin{bmatrix}1&-1\\0&1\\2&0\end{bmatrix}}\overset{w}{\begin{bmatrix}x\\y\end{bmatrix}} = \overset{v}{\begin{bmatrix}0\\1\\2\end{bmatrix}}$ have a solution?

If yes, then $\begin{bmatrix}0\\1\\2\end{bmatrix}$ is in $\operatorname{im}(A)$.

If no, then $\begin{bmatrix}0\\1\\2\end{bmatrix}$ is not in $\operatorname{im}(A)$.

Row reduce the augmented matrix:

$$\left[\begin{array}{cc|c}1&-1&0\\0&1&1\\2&0&2\end{array}\right] \xrightarrow[\text{swap } R_1,R_3]{} \left[\begin{array}{cc|c}2&0&2\\0&1&1\\1&-1&0\end{array}\right] \xrightarrow[R_1 \mapsto \frac{1}{2}R_1]{} \left[\begin{array}{cc|c}1&0&1\\0&1&1\\1&-1&0\end{array}\right] \xrightarrow[R_3 \mapsto -R_1+R_3]{} \left[\begin{array}{cc|c}1&0&1\\0&1&1\\0&-1&-1\end{array}\right] \xrightarrow[R_3 \mapsto R_2+R_3]{} \left[\begin{array}{cc|c}1&0&1\\0&1&1\\0&0&0\end{array}\right] \text{ in REF}$$

Note: We are only trying to answer the question

" Does $\begin{bmatrix}1&-1\\0&1\\2&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$ have a solution? "

Let's answer this question.

Recall: Once we have an augmented matrix in REF,
the original SLE has a solution (i.e. consistent) if and only if
the REF augmented matrix has no leading 1 in the right column.

$\left[\begin{array}{cc|c}1&0&1\\0&1&1\\0&0&0\end{array}\right]$ does <u>not</u> have a leading 1 in the right column,
so the SLE is consistent.

This means $\begin{bmatrix}1&-1\\0&1\\2&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$ has a solution.

So $\begin{bmatrix}0\\1\\2\end{bmatrix}$ is in $\operatorname{im}\left(\begin{bmatrix}1&-1\\0&1\\2&0\end{bmatrix}\right)$.

<u>Sanity check</u>

Actually solve $\begin{bmatrix}1&-1\\0&1\\2&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$

$\left[\begin{array}{cc|c}1&-1&0\\0&1&1\\2&0&2\end{array}\right] \xrightarrow[\text{Row reduce}]{} \left[\begin{array}{cc|c}1&0&1\\0&1&1\\0&0&0\end{array}\right]$ $\quad x = 1$, $y = 1$ $\quad$ Sol: $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

<u>Check:</u> $\begin{bmatrix}1&-1\\0&1\\2&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$ ✓

d) Does $A\begin{bmatrix}x\\y\end{bmatrix} = v$ have a solution? If yes, v is in $\mathrm{im}(A)$
If no, v is not in $\mathrm{im}(A)$

Does $\begin{bmatrix}1 & -1\\0 & 1\\2 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ have a solution?

Augmented matrix

$$\left[\begin{array}{cc|c}1 & -1 & 1\\0 & 1 & 0\\2 & 0 & 0\end{array}\right] \xrightarrow[\text{Swap } R_1, R_3]{} \left[\begin{array}{cc|c}2 & 0 & 0\\0 & 1 & 0\\1 & -1 & 1\end{array}\right] \xrightarrow[R_1 \mapsto \frac{1}{2}R_1]{} \left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\1 & -1 & 1\end{array}\right] \xrightarrow[R_3 \mapsto -R_1 + R_3]{} \left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & -1 & 1\end{array}\right] \xrightarrow[R_3 \mapsto R_2 + R_3]{} \left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right] \text{ in REF}$$

Recall: Once we have an augmented matrix in REF,
if this REF augmented matrix has a leading 1 in the right column,
we can conclude the original SLE has no solution (inconsistent).

$\left[\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{array}\right]$ has a leading 1 in the right column.
So the original SLE is inconsistent.

So $\begin{bmatrix}1 & -1\\0 & 1\\2 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$ has no solution.

Therefore, $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is <u>not</u> in $\mathrm{im}\left(\begin{bmatrix}1 & -1\\0 & 1\\2 & 0\end{bmatrix}\right)$.

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e) You can go through the same process to check whether $A\begin{bmatrix}x\\y\end{bmatrix} = v$ $\left(\text{where } v := \begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ has a solution.

But, by Thm 4, $\mathrm{im}(A)$ is a subspace of $\mathbb{R}^3$, and we've seen that a subspace contains the zero vector,
(see Note 1, slide 8/13)
So $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is in $\mathrm{im}(A)$.

Theorem 4: Images are subspaces

The image of an $m \times n$ matrix must be a subspace of $\mathbb{R}^m$.