Ch 6: Relations
Sec 6.1: Basic definitions

- ✓ I. Relations $A \to B$
- ✓ II. Relations on A
- ✓ III. Graph of a relation
- ✓ IV. Composition of relations

(Recall the definition of a cartesian product from Sec 1.3)

$A \times B = \{(a,b) \mid a \in A, b \in B\}$

I. Relations from a set into another

Ex $\{1,2,3\} \times \{2,3\}$ has $3 \cdot 2 = 6$ elements: $(1,2)$, $(1,3)$, $(2,2)$, $(2,3)$, $(3,2)$, $(3,3)$.

The (binary) relation "$<$" from $\{1,2,3\}$ into $\{2,3\}$ is the truth set for the proposition "$x < y$": (Sec 3.6)

$\{(1,2), (1,3), (2,3)\}$

This is because $1<2$ is true, $1<3$ is true, $2<2$ is FALSE, $2<3$ is true, $3<2$ is FALSE, $3<3$ is FALSE.

Def A relation r from a set A into a set B is any subset r of $A \times B$.

Ex • The relation "$<$" above is a relation from $\{1,2,3\}$ into $\{2,3\}$

• "$<$" is equal to $\{(1,2), (1,3), (2,3)\}$

Ex • A different relation from $\{1,2,3\}$ into $\{2,3\}$ is $\{(1,3), (3,2)\}$

• Another relation from $\{1,2,3\}$ into $\{2,3\}$ is $\{(2,3)\}$

• Yet another relation from $\{1,2,3\}$ into $\{2,3\}$ is the entire $\{1,2,3\} \times \{2,3\}$.

• The empty set is also a relation.

• How many relations are there from $\{1,2,3\}$ into $\{2,3\}$? Ans: 2^6. Why?

Relation notation: We write $x \, r \, y$ when $(x,y) \in r$

II. Relations on a set

Ex (Divisibility Example)

$A = \{2,3,5,6\}$. Define a relation r from A into A by $(a,b) \in r$ if and only if a divides evenly into b.

(i.e. a relation on A)

$a \neq 0$ and $\frac{b}{a}$ is an integer

Then $r = \{(2,2), (2,6), (3,3), (3,6), (5,5), (6,6)\}$

We write: $2\,r\,2$, $2\,r\,6$, $3\,r\,3$, $3\,r\,6$, $5\,r\,5$, $6\,r\,6$.

Def A relation on A is a relation r from A into itself, i.e. r is a subset of $A \times A$.

Ex Let $A = \{1,2,3\}$.

- The relation "divides" on A, as a subset of $A \times A$, is ...
  $\{(1,1), (1,2), (1,3), (2,2), (3,3)\}$
- The relation "equals" on A, as a subset of $A \times A$, is ...
  $\{(1,1), (2,2), (3,3)\}$
- The relation "is unequal to" on A, as a subset of $A \times A$, is ...
  $\{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$

Def Let $a, b \in \mathbb{Z}$, and $a \neq 0$. We say a divides b, denoted $a \mid b$, if there exists an integer k such that $ak = b$.

Ex The relation "divides" on $\mathbb{Z}$, as a subset of $\mathbb{Z} \times \mathbb{Z}$, is

$$\{\ldots, (-1,-1), (-1,0), (-1,1), (-1,2), (-1,3), \ldots, (1,-1), (1,0), (1,1), (1,2), (1,3), \ldots, (2,-2), (2,0), (2,2), (2,4), (2,6), \ldots, (3,-3), (3,0), (3,3), (3,6), (3,9), \ldots\} = \{(a,b) : a, b \in \mathbb{Z},\ a \neq 0,\ a \mid b\}$$

Q: Is $3 \mid 0$? Yes, so $(3,0) \in$ "divides"
Is $-2 \mid 6$? Yes, so $(-2,6) \in$ "divides"
Is $0 \mid 5$? No, so $(0,5) \notin$ "divides"

Book Exercise:

1. For each of the following relations r defined on $\mathbb{P}$, determine which of the given ordered pairs belong to r

(a) xry iff $x|y$; (2, 3), (2, 4), (2, 8), (2, 17) Ans: (2,4), (2,8)

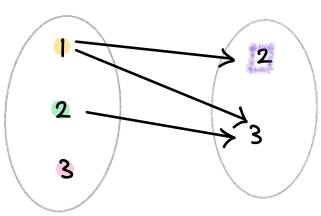
Do → (b) xry iff $x \leq y$; (2, 3), (3, 2), (2, 4), (5, 8) Ans: (2,3), (2,4), (5,8)

(c) xry iff $y = x^2$; (1,1), (2, 3), (2, 4), (2, 6) Ans: (1,1), (2,4)

III. Graph of a relation

Ex Below is a visualization for a relation r from $\{1,2,3\}$ into $\{2,3\}$

A $\xrightarrow{r}$ B



The three arrows indicate:
1 is related to 2 under r (1 r 2)
1 is related to 3 under r (1 r 3)
2 is related to 3 under r (2 r 3)

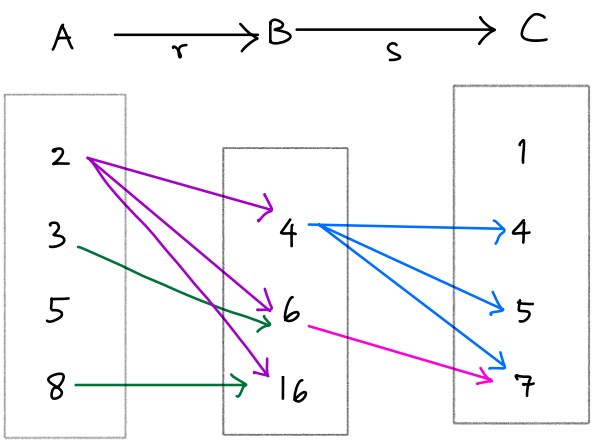
In this picture, the relation r, as a subset of $A \times B$, is the set $\{(1,2), (1,3), (2,3)\}$ from our first example "<"

IV. Composition of relations

Ex: Let $A=\{2,3,5,8\}$, $B=\{4,6,16\}$, $C=\{1,4,5,7\}$.

Let r be the relation "divides" from A into B, and let s be the relation $\leq$ from B into C

Visualize r and s using graphs.



Notice:

Some elements of A go through elements in B to C:

2 divides 4 and $4 \leq 4$

$2 \mid 4$ and $4 \leq 5$

$2 \mid 4$ and $4 \leq 7$

$2 \mid 6$ and $6 \leq 7$

note: there are two paths from 2 to 7, corresponding to the element (2,7)

$3 \mid 6$ and $6 \leq 7$

We can define a new relation rs from A into C this way.

As a subset of $A \times C$, $rs = \{(2,4), (2,5), (2,7), (3,7)\}$.

Def (Composition of relations)

Let r be a relation from A into B, and let s be a relation from B into C.

The composition rs is the set of pairs $(a,c) \in A \times C$ satisfying: $(a,c) \in rs$ if and only if

$$(a,b) \in r \text{ and } (b,c) \in s \text{ for some } b \in B.$$

Ex of composition of relations

$A = \{\text{UML students}\}$

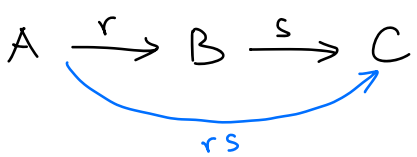
$B = \{\text{UML classes}\}$

$C = \{\text{UML profs}\}$

Let r be the "is enrolled in" relation from A into B ("$a\ r\ b$" means "student a is enrolled in class b")

Let s be the "is taught by" relation from B into C ("$b\ s\ c$" means "class b is taught by prof c").

Then the composite relation rs is the "has a teacher" relation from A into C
("$a\ rs\ c$" means "student a has prof c as a teacher")



Let $A = \{1,2\}$. Let r and s be relations from A to A given by $r = \{(1,1)\}$ and $s = \{(1,2)\}$. What is rs? What is sr? Are they equal?

..?..

Answer: $rs = \{(1,2)\}$; $sr = \{\}$ = the empty set. (See Sec 5.1 Matrix multiplication)

Compare: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

This is no coincidence! (Later the book talks about matrix representation of a relation)

3. (Extra) Let $A = \{1, 2, 3, 4, 5\}$ and define r on A by xry iff $x + 1 = y$. We define $r^2 = rr$ and $r^3 = r^2 r$. Find:

(a) $r = \{(1,2), (2,3), (3,4), (4,5)\}$

(b) $r^2 = \{(1,3), (2,4), (3,5)\} = \{(x,y) \in A \times A \mid y = x+2\}$

(c) $r^3 = \{(1,4), (2,5)\} = \{(x,y) \mid y - x = 3, \ x \in A, y \in A\}$

Note If r is a relation on A, we write $r^2 = rr$, $r^3 = r^2 r$, $r^4 = r^3 r$, etc.

Recap: A relation is a set of ordered pairs. rs means apply r first, then s.