Sec 5.2 Special types of matrices

Note: For all real numbers x, we have

$x + 0 = x$ "0 is an additive identity"

and

$x \cdot 1 = x$ "1 is a multiplicative identity"

Recall from Sec 5.1: the zero matrix $\vec{0}_{m \times n}$ is the $m \times n$ matrix whose entries are all 0.

For all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ acts a lot like the matrix version of the number 0, as far as addition of 2×2 matrices is concerned

You might then guess that the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ acts like the matrix version of the number 1, as far as multiplication of 2×2 matrices is concerned.

BUT, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a \cdot 1 + b \cdot 1 & a \cdot 1 + b \cdot 1 \\ c \cdot 1 + d \cdot 1 & c \cdot 1 + d \cdot 1 \end{pmatrix} = \begin{pmatrix} a+b & a+b \\ c+d & c+d \end{pmatrix}$

The matrix version of the number 1 is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

because $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the <u>2×2 identity matrix</u>,

denoted I_2 or I.

Def $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$ is called the <u>n×n identity matrix</u>

1's on the main diagonal,
0's everywhere else

Theorem $AI = A$ and $IA = A$ for all $A \in M_{n\times n}(\mathbb{R})$

n×n matrices

Def Let A be an n×n matrix.

- If there is an n×n matrix B such that $AB = I$ and $BA = I$, then B is an <u>inverse</u> of A and we say A is <u>invertible</u>

Thm Inverses are unique

Notation: A^{-1} is the inverse of A.

Ex • Compute $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ Ans: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So, $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ is the inverse of $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

• Compute $\begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ -\frac{1}{3} & \frac{1}{5} \end{pmatrix}$ Ans: $\begin{bmatrix} 1-\frac{2}{3} & \frac{1}{2}+\frac{2}{5} \\ -3-\frac{5}{3} & -\frac{3}{2}+1 \end{bmatrix} \neq I_2$

So these two matrices are <u>not</u> inverses

The determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$.

Ex $\det \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = 2 \cdot 3 = 6$

$\det \begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix} = 5 - (2)(-3) = 5 + 6 = 11$

$\det \begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix} = 6(-4) - (-3)8 = -24 + 24 = 0$

A formula for computing inverses of 2×2 matrices:

If $\det A \neq 0$, $A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

If $\det A = 0$, A^{-1} does not exist (A is not invertible)

Ex $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{-1} = \frac{1}{6} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix}^{-1} = \frac{1}{11} \begin{pmatrix} 5 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 5/11 & -2/11 \\ 3/11 & 1/11 \end{pmatrix}$

$\begin{pmatrix} 6 & -3 \\ 8 & -4 \end{pmatrix}$ doesn't have an inverse

Def A square matrix D is called a diagonal matrix if $d_{ij} = 0$ whenever $i \neq j$,

i.e. D can have any value on the main diagonal. Everywhere else, the entries of D are 0.

Ex. The identity matrix and zero square matrix are diagonal matrices

- $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ are diagonal
- $C = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{bmatrix}$ is not a diagonal matrix

Exercise 6 Prove by induction that

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \quad \text{for all } n = 1, 2, \ldots$$

Proof **Base case $n = 1$:** $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^1 & 0 \\ 0 & 3^1 \end{pmatrix}$ ✓

Induction Step: Suppose $k \geq 1$ and $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^k = \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix}$.

Then $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{k+1} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^k \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 2^k & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ by the Induction hypothesis

$= \begin{pmatrix} 2^k 2 & 0 \\ 0 & 3^k 3 \end{pmatrix}$

$= \begin{pmatrix} 2^{k+1} & 0 \\ 0 & 3^{k+1} \end{pmatrix}$ ☒

Exercise

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$, $C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Compute AB Ans: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Compute BA Ans: $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$

Compute det(A) Ans: 0

Compute det(B) Ans: 0

Compute det(C) Ans: $(2)(2) - (-1)(-1) = 4 - 1 = 3$

Find the inverse of A, if possible.

Find the inverse of B, if possible.

A and B are not invertible because $\det(A) = 0$ and $\det(B) = 0$

Find the inverse of C, if possible.

See the next page

2. For the given matrices A find A^{-1} if it exists and verify that $AA^{-1} = A^{-1}A = I$. If A^{-1} does not exist explain why.

(a) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$

(a) The determinant is (2)(2) – (–1)(–1) = 4 – 1 = 3.
So the desired matrix is

$$1/3 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

i.e.

$$\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$$

Check:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) No inverse exists since det $A = 0$. We can also prove by contradiction that the matrix does not have an inverse. For, suppose there exist w, x, y, z satisfying

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} y & z \\ 2y & 2z \end{pmatrix}$$

we would need to have

$$\begin{pmatrix} y & z \\ 2y & 2z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But this is impossible (do you see why?).