Ch 5 : Intro to Matrix Algebra

Sec 5.1 Basic Definitions & Operations

Def A <u>matrix</u> is a rectangular array of numbers of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix}$$

$\longleftarrow$ row 1 $\longrightarrow$

$\longleftarrow$ row m $\longrightarrow$

column 2 $\downarrow$

column 3 $\downarrow$

This matrix A has m rows and n columns, and we say the <u>size of A</u> (or order of A) is $m \times n$

For short, write $A = (a_{rc})_{1 \le r \le m,\ 1 \le c \le n}$

$M_{m\times n}(\mathbb{R}) = \{$ all $m \times n$ matrices whose entries are real numbers $\}$

$M_{m\times n}(\mathbb{Z}) = \{$ all $m \times n$ matrices whose entries are integers $\}$

Ex Let $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -5 & 1 \end{bmatrix}$, a 3×2 matrix

Then $a_{31} = -5$ (entry in row 3, col 1)

Let $B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$, a 2×2 matrix

Even though $b_{11} = a_{11}$, $b_{12} = a_{12}$

$b_{21} = a_{21}$, $b_{22} = a_{22}$

A and B are not equal

Def Two matrices A and B are called equal (notation: $A = B$) if:

(1) A and B have the same size

(2) All entries are equal, i.e. $a_{ij} = b_{ij}$ for all appropriate i and j.

Def (Matrix addition)

Let A and B be $m \times n$ matrices.

Then $A + B$ is the $m \times n$ matrix whose r,c entry is $a_{rc} + b_{rc}$

⚠ If two matrices have different sizes, $A+B$ is not defined

Ex $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -5 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 0 & 0 \end{bmatrix}$, $A + B = \begin{bmatrix} 1+1 & 0+0 \\ 3+3 & 2+2 \\ -5+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 4 \\ -5 & 1 \end{bmatrix}$

Def (Scalar multiplication)

"scalar" means "number"

Let $A = [a_{ij}]$ be an $m \times n$ matrix, and let s be a scalar

Then the scalar product sA is the $m \times n$ matrix

whose i,j-entry is $s a_{ij}$

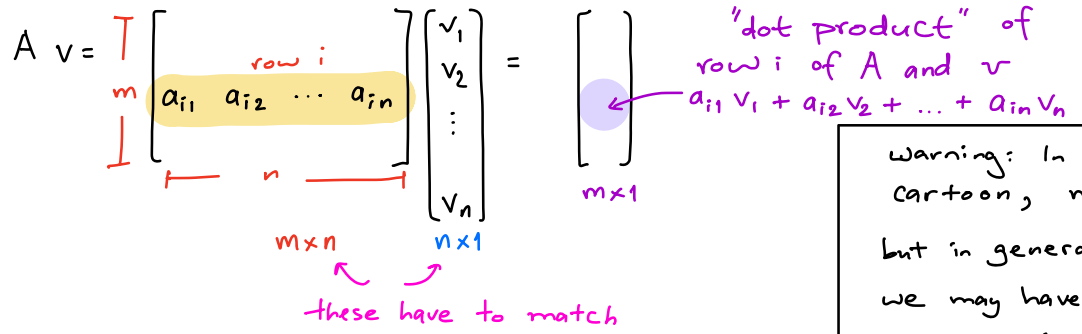
Ex $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -5 & 1 \end{bmatrix}$, $s = -2$, $sA = -2 \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -6 & -4 \\ 10 & -2 \end{bmatrix}$

Def (Subtraction)

If A, B are the same size, $A - B = A + (-1)B$

Def (Multiplying a matrix and a vector)

① Multiply an $m \times n$ matrix A with an $n \times 1$ column vector



warning: In this cartoon, $m < n$, but in general we may have $m = n$, $m < n$, $m > n$

② If width of A is not equal to height of v, then Av is undefined

Ex $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2×3 $\quad$ 3×1

$$Av = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + (-1) \cdot 2 + 2 \cdot 0 \\ 0 \cdot 1 + 1 \cdot 2 + 4 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2×3 $\quad$ 3×1 $\quad$ 2×1

$Aw = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not defined

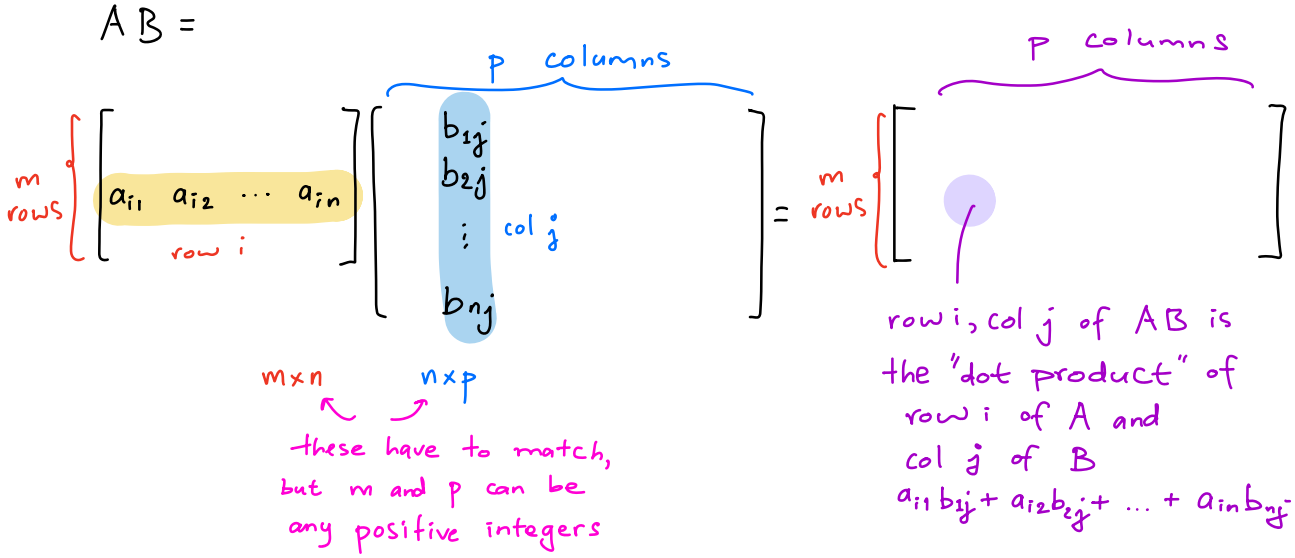
2×3 $\quad$ 2×1

do not match

Def (Matrix multiplication)

① Multiply an $m \times n$ matrix A with an $n \times p$ matrix B

$AB =$



② AB is defined if and only if width of A is equal to height of B

Ex $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix}$.

A: 2×3 $\quad$ B: 3×4

Since width of A = height of B, AB is the 2×4 matrix as defined above.

$$AB = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \end{bmatrix}$$

A · (2nd col of B) = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

v from previous example

$$AB = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} \cdot & 1 & 25 & \cdot \\ \cdot & 2 & \cdot & \cdot \end{bmatrix}$$

row 1, col 3 of AB = dot product of row 1 of A and col 3 of B

$= 3 \cdot 6 + (-1) \cdot 3 + 2 \cdot 5 = \boxed{25}$

$$AB = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} \cdot & 1 & 25 & \cdot \\ \cdot & 2 & \cdot & 36 \end{bmatrix}$$

row 2, col 4 of AB = dot product of row 2 of A and col 4 of B

$= 0 \cdot 0 + 1 \cdot 4 + 4 \cdot 8 = \boxed{36}$

Now compute the other four entries of AB

Answer:

$$AB = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & 2 & 3 & 4 \\ -1 & 0 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 25 & 12 \\ -4 & 2 & 23 & 36 \end{bmatrix}$$

Question? What is BA?

Answer Not defined! B A

$3 \times 4 \quad 2 \times 3$

these do not match

Def • An $n \times n$ matrix is called a square matrix.

• If A is square, $A^2 = AA$ is defined, $A^3 = AAA$ is defined, etc.

Ex $A = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$, $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$, $A^3 = \begin{pmatrix} -1 & 0 \\ 0 & 3^3 \end{pmatrix}$, $A^4 = \begin{pmatrix} (-1)^4 & 0 \\ 0 & 3^4 \end{pmatrix}$

Note: This is an example of a diagonal matrix.

Homework (See Sec 5.2 Exercise 6)

Prove by induction that $\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} (-1)^n & 0 \\ 0 & 3^n \end{pmatrix}$ for all positive integers n

Def The $m \times n$ matrix whose entries are all 0 is denoted $\vec{0}_{m \times n}$ or $\vec{0}$ or 0 or zero

Ex: $\vec{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Exercise

Calculate the matrix sum $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$

Calculate the scalar product $5\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} 30 & 35 \\ 40 & 45 \end{pmatrix}$

Calculate the matrix product $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 11 \\ 13 & 18 \end{pmatrix}$

$= \left(\begin{array}{c|c} 1\cdot 2 + 2\cdot 3 & 1\cdot 3 + 2\cdot 4 \\ \hline 2\cdot 2 + 3\cdot 3 & 2\cdot 3 + 3\cdot 4 \end{array}\right)$

$= \left(\begin{array}{c|c} 2+6 & 3+8 \\ \hline 4+9 & 6+12 \end{array}\right)$

Longer exercises

Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 3 & -5 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$

1.  Compute $A+B$,  Compute $B+A$

2.  Compute AB,  Compute BA

3.  Compute $A(BC)$,  Compute $(AB)C$

Longer exercises

Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 3 & -5 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}$

① Compute $A+B$, ~~Compute $B+A$~~

Ans: $A+B = B+A = \begin{pmatrix} 1 & 0 \\ 5 & -2 \end{pmatrix}$

② Compute AB, ~~Compute BA~~

Ans: $AB = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 3 & -5 \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ 9 & -13 \end{pmatrix}$

$BA = \begin{pmatrix} 0 & 1 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -7 & -18 \end{pmatrix}$

Matrix multiplication is not commutative

③ Compute $A(BC)$, ~~Compute $(AB)C$~~

Ans $\left(\begin{array}{c|c} 18 & -15 \\ \hline -39 & 35 \end{array}\right)$

Matrix multiplication is associative, so we can simply write ABC