4.2 Laws of Set Theory

4.2.1 Tables of Laws

The following basic set laws can be derived using either the Basic Definition or the Set-Membership approach and can be illustrated by Venn diagrams.

Table 4.2.1 Basic Laws of Set Theory

Commutative Laws	
(1) $A \cup B = B \cup A$	(1′) $A \cap B = B \cap A$
Associative Laws	
(2) $A \cup (B \cup C) = (A \cup B) \cup C$	(2′) $A \cap (B \cap C) = (A \cap B) \cap C$
Distributive Laws	
(3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	(3′) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Identity Laws	
(4) $A \cup \emptyset = \emptyset \cup A = A$	(4′) $A \cap U = U \cap A = A$
Complement Laws	
(5) $A \cup A^c = U$	(5′) $A \cap A^c = \emptyset$
Idempotent Laws	
(6) $A \cup A = A$	(6′) $A \cap A = A$
Null Laws	
(7) $A \cup U = U$	(7′) $A \cap \emptyset = \emptyset$
Absorption Laws	
(8) $A \cup (A \cap B) = A$	(8′) $A \cap (A \cup B) = A$
DeMorgan's Laws	
(9) $(A \cup B)^c = A^c \cap B^c$	(9′) $(A \cap B)^c = A^c \cup B^c$
Involution Law	
(10) $(A^c)^c = A$	

4.2.2 Proof Using Previously Proven Theorems

Once a few basic laws or theorems have been established, we frequently use them to prove additional theorems. This method of proof is usually more efficient than that of proof by Definition. To illustrate, let us prove the following Corollary to the Distributive Law. The term "corollary" is used for theorems that can be proven with relative ease from previously proven theorems.

Corollary 4.2.2 A Corollary to the Distributive Law of Sets. *Let A and B be sets. Then $(A \cap B) \cup (A \cap B^c) = A$.*

Proof.

$$
\begin{aligned}
(A \cap B) \cup (A \cap B^c) &= A \cap (B \cup B^c) \\
&\quad \text{Why? By the 1st Distributive Law of sets (3)} \\
&= A \cap U \\
&\quad \text{Why? By the 1st Complement Law (5)} \\
&= A \\
&\quad \text{Why? By the 2nd Identity Law of sets (4')}
\end{aligned}
$$

4.2.3 Proof Using the Indirect Method/Contradiction

Theorem 4.2.3 An Indirect Proof in Set Theory. *Let A, B, C be sets. If $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.*

Proof

Assume $A \subseteq B$ and $B \cap C = \emptyset$, and $A \cap C \neq \emptyset$. To prove that this cannot occur, let $x \in A \cap C$. (second premise)

$$
\begin{aligned}
x \in A \cap C &\Rightarrow x \in A \text{ and } x \in C \\
&\Rightarrow x \in B \text{ and } x \in C. \\
&\Rightarrow x \in B \cap C
\end{aligned}
$$

But this contradicts the second premise. Hence, the theorem is proven. ■

Group work: 4.2:4(b) (5 minutes)

Use the laws of set theory and previously proved theorems to prove Claim:

$$A \cap (B \cap (A \cap B)^c) = \varnothing$$

Possible Proof 1

..?..

$$\begin{aligned}
&A \cap (B \cap (A \cap B)^c) \\
&\quad = A \cap (B \cap (A^c \cup B^c)) && \text{(De Morgan)} \\
&\quad = A \cap (B \cap A^c \cup B \cap B^c) && \text{(distributive law)} \\
&\quad = A \cap (B \cap A^c \cup \varnothing) && \text{(complement law)} \\
&\quad = A \cap (B \cap A^c) && \text{(identity law)} \\
&\quad = A \cap (A^c \cap B) && \text{(commutative law)} \\
&\quad = (A \cap A^c) \cap B && \text{(associative law)} \\
&\quad = \varnothing \cap B && \text{(complement law)} \\
&\quad = \varnothing && \text{(null law)}
\end{aligned}$$

Possible Proof 2

$$\begin{aligned}
&A \cap (B \cap (A \cap B)^c) \\
&\quad = (A \cap B) \cap (A \cap B)^c && \text{(associative law)} \\
&\quad = \varnothing && \text{(complement law applied to the set } A \cap B\text{)}
\end{aligned}$$

Exercise 4.2 : 4(b)

Use the laws of set theory and previously proved theorems to prove Claim:

$$A \cap (B \cap (A \cap B)^c) = \varnothing$$

Possible Proof 3

Alternatively, here's a paragraph-style proof:

Proof by contradiction: Suppose $A \cap (B \cap (A \cap B)^c)$ is non-empty.

Let $x \in A \cap (B \cap (A \cap B)^c)$.

Then $x \in A$ and $x \in B \cap (A \cap B)^c$.

Since $x \in B \cap (A \cap B)^c$, $x \in B$ and $x \in (A \cap B)^c$.

Thus we have $x \in A$, $x \in B$, and $x \in (A \cap B)^c$.

But $x \in A$ and $x \in B$ imply $x \in A \cap B$, contradicting $x \in (A \cap B)^c$.

Contradiction!

Hence $A \cap (B \cap (A \cap B)^c)$ must be empty.

—end—