4.1 Methods of Proof for Sets

Definition 4.1.1 Counterexample. An example that disproves a statement is called a counterexample. ◆

Example 4.1.2 Disproving distributivity of addition over multiplication. From basic algebra we learned that multiplication is distributive over addition. Is addition distributive over multiplication? That is, is $a + (b \cdot c) = (a + b) \cdot (a + c)$ always true? If we choose the values $a = 3$, $b = 4$, and $c = 1$, we find that $3+(4\cdot 1) \neq (3+4)\cdot(3+1)$. Therefore, this set of values serves as a counterexample to a distributive law of addition over multiplication.

Theorem 4.1.7 The Distributive Law of Intersection over Union. *If A, B, and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.*

Proof. What we can assume: A, B, and C are sets.

What we are to prove: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

We must prove: (a) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Let $x \in A \cap (B \cup C)$:

$x \in A \cap (B \cup C) \Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

def. of union and intersection

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

distributive law of logic

$\Rightarrow (x \in A \cap B)$ or $(x \in A \cap C)$

def. of intersection

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

def. of union

We must also prove (b) $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

$x \in (A \cap B) \cup (A \cap C) \Rightarrow (x \in A \cap B)$or $(x \in A \cap C)$

Why?

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

Why?

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

Why?

$\Rightarrow x \in A \cap (B \cup C)$

Why? □

Claim Let A, B, and C be sets. Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof Assume $x \in A$.

Since $x \in A$ and $A \subseteq B$, $x \in B$.

Since $x \in B$ and $B \subseteq C$, $x \in C$.

So every element of A is an element of C.
Hence $A \subseteq C$.

Ex 1 Group work: Exercise 4.1:1(c) (4 minutes)
Let A, B, and C be sets. Prove that if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

..?..

Proof Assume $x \in A$.

Since $A \subseteq B$ and $A \subseteq C$, $x \in B$ and $x \in C$.

Hence $x \in B \cap C$.

So every element of A is an element of $B \cap C$.
So $A \subseteq B \cap C$.

Ex 2 True or false? "If $A \subseteq C$, then $A \subseteq B$ and $B \subseteq C$."

Ans: False. A counterexample is
$A = \{1, 2\}$, $B = \{6, 7, 8\}$, $C = \{1, 2, 3\}$

(Extra Example)

Ask: What is the converse of "If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$"?

..?..

The converse is "If $A \subseteq B \cap C$, then $A \subseteq B$ and $A \subseteq C$."

Is it true or false ?

The converse is true.

Proof: Let $x \in A$.

Since $x \in A$ and $A \subseteq B \cap C$, $x \in B \cap C$.

So $x \in B$ and $x \in C$.

Since every element of A is an element of B and every element of A is an element of C, $A \subseteq B$ and $A \subseteq C$.

Extra Example

Theorem 4.1.8: $A \times (B \cap C) = A \times B \ \cap \ A \times C$.

Discuss proof:

(a) $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.
Let $(x, y) \in A \times (B \cap C)$.

$$
\begin{aligned}
(x, y) \in A \times (B \cap C) &\Rightarrow x \in A \text{ and } y \in (B \cap C) \\
&\quad \text{Why?} \\
&\Rightarrow x \in A \text{ and } (y \in B \text{ and } y \in C) \\
&\text{Why?} \\
&\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\
&\quad \text{Why?} \\
&\Rightarrow (x, y) \in (A \times B) \text{ and } (x, y) \in (A \times C) \\
&\quad \text{Why?} \\
&\Rightarrow (x, y) \in (A \times B) \cap (A \times C) \\
&\quad \text{Why?}
\end{aligned}
$$

(b) $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.
Let $(x, y) \in (A \times B) \cap (A \times C)$.

$$
\begin{aligned}
(x, y) \in (A \times B) \cap (A \times C) &\Rightarrow (x, y) \in A \times B \text{ and } (x, y) \in A \times C \\
&\quad \text{Why?} \\
&\Rightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\
&\quad \text{Why?} \\
&\Rightarrow x \in A \text{ and } (y \in B \text{ and } y \in C) \\
&\quad \text{Why?} \\
&\Rightarrow x \in A \text{ and } y \in (B \cap C) \\
&\quad \text{Why?} \\
&\Rightarrow (x, y) \in A \times (B \cap C) \\
&\quad \text{Why?}
\end{aligned}
$$