Sec 3.7 A review of methods of proofs

Def

An integer m is called <u>odd</u> $\Leftrightarrow m = 2j+1$ for some $j \in \mathbb{Z}$.

$(\exists j)_{\mathbb{Z}} \; (m = 2j+1)$

An integer n is called <u>even</u> $\Leftrightarrow n = 2k$ for some $k \in \mathbb{Z}$.

$\exists\, k \in \mathbb{Z}$ such that $n = 2k$

Q: What is the difference between a direct proof of $P \Rightarrow Q$ and an indirect proof?

→ Direct proof: To prove $P \Rightarrow Q$,

assume P is true and prove Q is true.

→ Indirect proof: To prove $P \Rightarrow Q$,

assume P and $\neg Q$ are true and derive a contradiction.

Example of a direct proof (that the sum of an odd number and an even number must be odd)

Claim: If m is odd and n is even, $m+n$ is odd

<u>Proof</u> Suppose m is odd and n is even.

Then we can write $m = 2j+1$ for some $j \in \mathbb{Z}$,

and we can write $n = 2k$ for some $k \in \mathbb{Z}$.

Then $m+n = (2j+1) + (2k) = 2(j+k) + 1$,

so $m+n$ is odd $\square$

Example of a direct proof (that the product of two odd integers must be odd).

Theorem: The product of two odd integers is an odd integer.

Proof: Let m and n be odd integers. Since m and n are odd, we can write $m=2j+1$ and $n=2k+1$ for integers j,k. But then $mn = (2j+1)(2k+1) = 4jk+2j+2k+1 = 2(2jk+j+k)+1$, so mn is odd.

(ended here Week 9 Tue)

Exercise 1: Prove that the sum of two odd integers must be even.

Exercise 2: prove that the product of two even integers is even.

Exercise 3: If m^2+7 is even then m is odd

Exercise 4: Suppose x and y are real numbers. If $x+y \leq 1$, then $x \leq \frac{1}{2}$ or $y \leq \frac{1}{2}$

(Sol)

Exercise1: Prove that the sum of two odd integers must be even.

Theorem: The sum of two odd integers is an even integer.

Proof: Let m and n be odd integers. Since m and n are odd, we can write $m=2j+1$ and $n=2k+1$ for integers j,k. But then $m+n = (2j+1) + (2k+1) = 2(j+k+1)$, so $m+n$ is even.

Exercise2: prove that the product of two even integers is even.

Theorem: If m and n are even, then mn is even.

Proof: Let m and n be even integers.
Since m and n are even,
we can write $m = 2j$ and $n = 2k$
for some integers j, k.
Then $mn = (2j)(2k)$
$= 2(2jk)$
So mn is even.

(sol)

Example of an indirect proof

Claim: If m^2+7 is even then m is odd

Proof Assume m^2+7 is even. For the sake of contradiction, suppose that m is not odd. Then m is even. Thus m^2 is even (since the product of two even integers is always even). The fact that m^2 is even and that 7 is odd tells us that m^2+7 is odd (since the sum of an even number and an odd number is always odd). This contradicts our assumption that m^2+7 is even. □

Example of a proof using the contrapositive

Claim: Suppose x and y are real numbers. If $x+y \leq 1$, then $x \leq \frac{1}{2}$ or $y \leq \frac{1}{2}$

Proof We will prove the contrapositive:

If "$x \leq \frac{1}{2}$ or $x \leq \frac{1}{2}$" is false, then "$x+y \leq 1$" is false

(i.e. if $x > \frac{1}{2}$ and $y > \frac{1}{2}$ then $x+y > 1$)

Suppose $x > \frac{1}{2}$ and $y > \frac{1}{2}$. Adding these inequalities, we get $x+y > \frac{1}{2}+\frac{1}{2} = 1$. □

Theorem: The square root of 2 is irrational. (Textbook exercise)

Proof: By contradiction. Suppose sqrt(2) were rational; say sqrt(2) = p/q for positive integers p,q.

Suppose also that p and q have no common factors > 1.

Since sqrt(2) = p/q, we get $2 = p^2 / q^2$, so $\mathbf{2\,q^2 = p^2}$.

Look at the possibilities for the last digit of p, and the associated last digit of p^2:

p	=	…0	…1	…2	…3	…4	…5	…6	…7	…8	…9
p^2	=	…0	…1	…4	…9	…6	…5	…6	…9	…4	…1

Likewise, look at the possibilities for q and $2q^2$:

q	=	…0	…1	…2	…3	…4	…5	…6	…7	…8	…9
q^2	=	…0	…1	…4	…9	…6	…5	…6	…9	…4	…1
$2q^2$	=	…0	…2	…8	…8	…2	…0	…2	…8	…8	…2

The last digit of p^2 must be 0, 1, 4, 5, 6, or 9, while the last digit of $2q^2$ must be 0, 2, or 8.

The only way p^2 can equal $2q^2$ is if both of them end in 0.

But then p and q are both multiples of 5, contradicting our assumption that p and q have no common factor.

This contradiction shows that our assumption (the rationality of the square root of 2) is untenable.

Therefore the square root of 2 is irrational.

Exercise: Use the same method to prove that the square root of 3 is irrational

(Textbook exercise)

Theorem: The square root of 3 is irrational.

Proof: By contradiction. Suppose sqrt(3) were rational; say sqrt(3) = p/q for positive integers p,q.

Suppose also that p and q have no common factors > 1.

Since sqrt(3) = p/q, we get $3 = p^2 / q^2$, so $\mathbf{3\,q^2 = p^2}$.

Look at the possibilities for the last digit of p, and the associated last digit of p^2:

p	=	…0	…1	…2	…3	…4	…5	…6	…7	…8	…9
p^2	=	…0	…1	…4	…9	…6	…5	…6	…9	…4	…1

Likewise, look at the possibilities q and $3q^2$:

q	=	…0	…1	…2	…3	…4	…5	…6	…7	…8	…9
q^2	=	…0	…1	…4	…9	…6	…5	…6	…9	…4	…1
$3q^2$	=	…0	…3	…2	…7	…8	…5	…8	…7	…2	…3

The last digit of p^2 must be 0, 1, 4, 5, 6, or 9, while the last digit of $3q^2$ must be 0, 2, 3, 5, 7, or 8.

The only way p^2 can equal $3q^2$ is if both of them end in 0 or 5.

But then p and q are both multiples of 5, contradicting our assumption that p and q have no common factor.