Sec 3.8 Quantifiers

Recall def from Sec 3.6:

$p(n)$ is a proposition over a universe U

The truth set of $p(n)$ is $T_p = \{ n \in U \mid p(n) = \text{true} \}$,

a subset of U.

We're often concerned with ...

(I) Whether T_p is empty or not (i.e. whether p is a contradiction or not)

(II) Whether $T_p = U$ or $T_p \neq U$ (i.e. whether p is a tautology or not)

Part I. The Existential Quantifier $\exists$

Def If T_p is not empty, we say/write:

"There exists an n in U such that $p(n)$ (is true)" or

"There is at least one n belonging to U satisfying $p(n)$" or

"There exists some n in U for which $p(n)$ is satisfied" or

(mix and match these English expressions)

In symbols, we write:

$(\exists n)_U \ (p(n))$ or $(\exists n)\ (p(n))$ if the context is clear or

$(\exists n \in U)\ (p(n))$

$\exists n \in U$ such that $p(n)$

$p(n)$ holds for some $n \in U$

Ex 1

(a) The symbol $(\exists k)_{\mathbb{Z}}\ (k^2+5k+6=0)$ means

"there is an integer that solves the equation $k^2+5k+6=0$"

Since there are two integers, $k=-2$ and $k=-3$,

that solve the equation $k^2+5k+6=0$,

the proposition $(\exists k)_{\mathbb{Z}}\ (k^2+5k+6=0)$ is true

(b) $(\exists k)_{\mathbb{Z}}\ (3k=12)$ means "12 is a multiple of 3"

which is true

(b) $(\exists k)_{\mathbb{Z}}\ (3k=10)$ means "10 is a multiple of 3"

which is false

(c) $(\exists r)_{\mathbb{Q}}\ (r^2=2)$ means "there is a rational number whose square is 2" or "$\sqrt{2}$ is a rational number"

which is false.

(d) Let $D(x)$: "x was born in December", $U=\{$persons in this room$\}$

$\exists\, x \in U$ such that $D(x)$

means "Someone in this room was born in December"

Part II. The Universal Quantifier $\forall$

Def If the truth set of the proposition $p(n)$ over the universe U is $T_p = U$, we say/write:

"For all n in U, $p(n)$ (is true)" or

"Every n belonging to U satisfies $p(n)$" or

"For each n of U, the property $p(n)$ holds" or

"Any n in U satisfies $p(n)$" or

(mix and match these English expressions)

In symbols, we write:

$(\forall n)_U \ (p(n))$ or $(\forall n)\ (p(n))$ or if the context is clear

$(\forall n \in U)(p(n))$

$\forall n \in U,\ p(n)$

$p(n),\ \forall n \in U$

$p(n)$ is true for all $n \in U$

Ex 2 (a) $(\forall x)_{\mathbb{R}} \ (x^2 \geq 0)$ means

"the square of every real number is non-negative"

which is true

(b) $(\forall n)_{\mathbb{Z}} \ (n+0=n$ and $0+n=n)$ means

"the sum of zero and any integer n is n" which is true

(c) The symbol $(\forall k)_{\mathbb{Z}} \ (k^2+5k+6=0)$ means

"Every integer solves the equation $k^2+5k+6=0$"

which is false

(d) Let $C(x)$: "x is a critic", $U = \{\text{humans}\}$

$\forall x \in U$, $C(x)$ means

"Everybody's a critic"

Part III The negation of quantified propositions

Ex 3 Let $U = \{\text{animals}\}$, $F(x)$: x is a fish,

$W(x)$: x lives in water.

Let $p(x)$ be the proposition $W(x) \to F(x)$,

so $p(x)$: "If x lives in water then x is a fish"

We know the truth set $T_p \neq U$, for example dolphins and lobsters are in U but not in T_p,

so

$$\forall x \in U, \quad W(x) \to F(x)$$

is false.

i.e. $\neg\left(\forall x \in U, \quad W(x) \to F(x)\right)$

is true.

i.e. $\exists x \in U$ such that $\neg(W(x) \to F(x))$

premise is true ($W(x)$) and conclusion is false ($F(x)$)

i.e. $\exists x \in U$ such that $W(x) \wedge \neg F(x)$

"there exist (at least one) an animal that lives in water and is not a fish"

De Morgan's Laws for quantifiers

(General rule for negation of quantified expressions)

(1) (Negation of a universally quantified proposition is an existentially quantified proposition)

$$\neg(\forall n \in U,\ p(n)) \iff \exists n \text{ such that } \neg p(n)$$

(2) (Negation of an existentially quantified proposition is a universally quantified proposition)

$$\neg(\exists n \in U \text{ such that } p(n)) \iff \forall n \in U,\ \neg p(n)$$

Ex 4 What is the negation of "everybody's a critic"?

Warning The answer is not "nobody's a critic"

- To see why not, consider the universe $U = \{\text{persons in this classroom}\}$.
- Let $U_1 = \{\text{persons in this classroom wearing glasses}\}$, $U_2 = \{\text{persons in this classroom not wearing glasses}\}$.
- Suppose every x in U_1 is a critic and every x in U_2 is not a critic.
- Then "everybody's a critic" is false (since people not wearing glasses are not critic) and "nobody's a critic" is also false (since people wearing glasses are critics).

- But the negation of a proposition and the original proposition cannot have the same truth value.

Correct answer: Let $C(x)$: x is a critic.

"everybody's a critic" translated into symbols is $\forall x \in U, \quad C(x)$

So the negation is

$\exists x \in U$ such that $\neg C(x)$

which is "there exists (at least) one who is not a critic"

Ex The negation of

"Someone in this room was born in December"

$\exists x \in U, D(x)$

is

"No one in this room was born in December"

$\forall x \in U, \neg D(x)$

Ex Saying $\sqrt{2}$ is not a rational number in symbols is

$\neg(\exists r \in \mathbb{Q}, \quad r^2 = 2)$.

This is equivalent to

$\forall r \in \mathbb{Q}, \quad r^2 \neq 2$.

Exercises ⇩

Problem I

Let $C(x)$: x is cold-blooded,
$F(x)$: x is a fish
$S(x)$: x lives in the sea

(1) Translate into English:

$$\forall x, \quad F(x) \longrightarrow S(x)$$

(2) Translate into English:

$$\exists x, \quad S(x) \wedge \neg F(x)$$

(3) Translate into a formula:
Every fish is cold-blooded

Problem II

Write the negation of

$$\forall n \in \mathbb{P}, \; n^2 - n + 41 \text{ is prime}$$

as an existentially quantified proposition

Problem III

What is the negation of

"Everybody's somebody's fool"?

Exercise (#1)

Let $C(x)$: x is cold-blooded,

$F(x)$: x is a fish

$S(x)$: x lives in the sea

(1) Translate into English:

$$\forall x, \quad F(x) \longrightarrow S(x)$$

Ans: All fish live in the sea

(2) Translate into English:

$$\exists x, \quad S(x) \wedge \neg F(x)$$

Ans: There's a sea-dwelling creature that isn't a fish

(3) Translate into a formula:

Every fish is cold-blooded

Ans: $\forall x, \quad F(x) \longrightarrow C(x)$

OR $\forall x \in T_F, \quad C(x)$

Write the negation of $\forall n \in \mathbb{P}$, n^2-n+41 is prime as an existentially quantified proposition

Ans: $\exists n \in \mathbb{P}$, n^2-n+41 is prime

by De Morgan's laws for quantifiers

What is the negation of

"Everybody's somebody's fool"?

Ans: Let $F(x,y)$: "x is y's fool".

"Everybody's somebody's fool" is

$$\forall x, \exists y, F(x,y).$$

Let's negate it!

$$\neg[\forall x, \exists y, F(x,y)]$$

By De Morgan's law for quantifiers, this becomes

$$\exists x, \neg(\exists y, F(x,y))$$

By De Morgan's law for quantifiers, this becomes

$$\exists x, \forall y, \neg F(x,y)$$

In English: "There exist some person x such that, for all persons y, x is not y's fool"

or

"There exists someone who is nobody's fool"

or

"Somebody's nobody's fool"