Sec 3.7 Mathematical Induction

Warm-up:

* Detachment law (from Sec 3.4 table):

$$(P \text{ and } P \to Q) \implies Q \text{ and } P$$

* Longer chains of detachment:

$$(P \text{ and } P \to Q \text{ and } Q \to R) \implies R \text{ and } Q \text{ and } P$$

* Even longer:

$$(P_1 \text{ and } P_1 \to P_2 \text{ and } P_2 \to P_3 \text{ and } P_3 \to P_4) \implies P_1 \wedge P_2 \wedge P_3 \wedge P_4$$

* Even longer:

$$P_1 \wedge (P_1 \to P_2) \wedge (P_2 \to P_3) \wedge \dots \wedge (P_{n-1} \to P_n) \text{ implies } P_1 \wedge P_2 \wedge \dots \wedge P_n.$$

$$P_1 \wedge \left(\bigwedge_{k=1}^{n-1} (P_k \to P_{k+1}) \right) \text{ implies } \bigwedge_{k=1}^{n} P_k$$

* The principle of mathematical induction is the law of logic that extends this by sending n to infinity:

The infinite conjunction

$$P_1 \wedge (P_1 \to P_2) \wedge (P_2 \to P_3) \wedge \dots \text{ (forever)}$$

implies the infinite conjunction

$$P_1 \wedge P_2 \wedge P_3 \wedge \dots \text{ (forever)}$$

i.e. $$P_1 \wedge \left(\bigwedge_{k=1}^{\infty} (P_k \to P_{k+1}) \right) \text{ implies } \bigwedge_{k=1}^{\infty} P_k$$

* What is this law of logic good for?

A: If you can prove P_1 and you can prove the infinitely many propositions $P_k \to P_{k+1}$ in some uniform manner, then you can conclude that $P_1, P_2, \ldots$ are all true.

Principal of mathematical induction

If

P_1 is true,

$P_1 \Rightarrow P_2$,

$P_2 \Rightarrow P_3$,

$P_3 \Rightarrow P_4$,

etc

P_1 is called the "base case"

for all $k \geq 1$ $P_k \Rightarrow P_{k+1}$

called the "induction step"

P_k is called the "induction hypothesis" and P_{k+1} the "induction conclusion"

then P_n is true for all $n = 1, 2, 3, \ldots$

Ex / Exercise (#5)

Use mathematical induction to show that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all $n \geq 1$.

Proof Base case $n=1$: $\frac{1}{1\cdot 2} = \frac{1}{(1+1)}$ ✓

Induction step: Suppose

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

This is the "inductive hypothesis"

for some $k \geq 1$.

Then

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

(by the inductive hypothesis)

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

By the principal of mathematical induction, the equality holds for all $k \geq 1$ □

Ex/ Exercise: Use mathematical induction to show that

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

for all $n \geq 1$.

Proof: Let P_n be the proposition

"$1 + 2 + \dots + n = \frac{n(n+1)}{2}$"

Warning: P_n is not the expression $\frac{n(n+1)}{2}$,
P_n is not a number,
P_n is a proposition

Base case $n = 1$: $1 = \frac{1(1+1)}{2}$ is true

P_1 is true ✓

Induction step (proving $P_k \Rightarrow P_{k+1}$):

Assume $k \geq 1$ and that P_k is true:

$$1 + 2 + \dots + k = \frac{k(k+1)}{2}$$

(Now we need to show $1 + 2 + \dots + k + (k+1) = \frac{(k+1)(k+1+1)}{2}$)

$$1 + 2 + \dots + k + (k+1) = \frac{k(k+1)}{2} + k + 1 \quad \text{by the induction hypothesis } (P_k)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

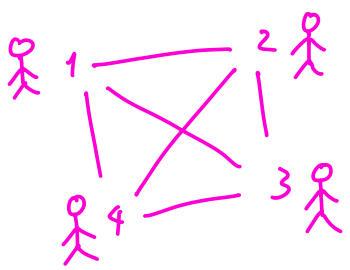
$$= \frac{(k+1)((k+1)+1)}{2}$$

so P_{k+1} is also true.

By the principal of mathematical induction, P_n is true for all $n \in \mathbb{P}$ (positive integers) □

Exercise (#9)

Within your group, shake everybody's hand exactly once. How many handshakes happened?



For a group of size 4, there are Six handshakes

Use mathematical induction to show that $\frac{(n-1)n}{2}$ handshakes will occur in a group of n people, for $n \geq 2$.

Proof **Base case** $n=2$: $\frac{(2-1)2}{2} = 1$ handshake ✓

Induction step:

- Assume, for some $k \geq 2$, $\frac{(k-1)k}{2}$ handshakes happen.
- One more person joins the group and shakes hands with k people, so the total number of handshakes for a group of $k+1$ people is

$$\frac{(k-1)k}{2} + k = \frac{k^2-k+2k}{2}$$

$$= \frac{k^2+k}{2}$$

$$= \frac{k(k+1)}{2}$$

$$= \frac{((k+1)-1)(k+1)}{2}$$

□

Exercise (#10) (Extra)

Use mathematical induction to show that we can make up any postage of eight cents or more using only 3- and 5-cent stamps.

Proof

Base case $n=8$:

Use one 3-cent stamp and one 5-cent stamp.

Inductive step:

Assume we can make k using only 3- and 5-cent stamps.

Case I: At least one 5-cent stamp is used in making n cents. In this case, replace this 5-cent with two 3-cent stamps. We now have $k+1$ cents.

Case II: Otherwise, no 5-cent stamps are used in making n cents. Then only 3-cent stamps are used. Thus at least three 3-cent stamps are used, since $n \geq 8$ which is larger than 6. Replace these three 3-cent stamps with two 5-cent stamps. We now have $k+1$ cents. □

— end —