Sec 3.6 Propositions over a universe

HW due tonight (can resubmit)
Thurs quiz: Sec 3.2, 3.3, 3.4

Def Let U be a nonempty (universe) set.

① A <u>proposition over the universe U</u> is a function

$$p: U \longrightarrow \{\text{true}, \text{false}\}$$

i.e. p is a sentence that takes in as input any value x in U and p has a truth value for every x in U.

② The <u>truth set</u> of p is

$$T_p = \{ a \in U \mid p(a) = \text{true} \}$$

Recall: A <u>prime</u> is an integer bigger than 1 whose only divisors are 1 and itself.

Ex Let $p(x)$ be the proposition "x is prime", $U = \{2,3,4\}$

Then p is an example of a proposition over $U = \{2,3,4\}$.

Since U is finite, we can represent p by the table

x	$p(x)$	
2	1	True
3	1	True
4	0	(false)

Then the truth set of p is $T_p = \{2, 3\}$

If we take the same p over a universe $\{1, 2, \ldots, 10\}$, then the truth set of p is $T_p = \{2, 3, 5, 7\}$.

Ex Let $U = \mathbb{P}$, the set of all positive integers

① The truth set of "$0 \le n \le 5$" is $\{1, 2, 3, 4, 5\}$
(↑ variable n)

② The truth set of "k is a multiple of 3" is
(↑ variable k)

$$\{3x \mid x \in \mathbb{P}\} = \{3, 6, 9, 12, \ldots\}$$

Ex Let U be the power set $\mathcal{P}(\{1,2,3,4\})$
i.e. { subsets of $\{1,2,3,4\}$ }

Let the proposition f be "$\{1,2\} \cap A = \varnothing$" over U.

$f(\varnothing)$ = true, since $\{1,2\} \cap \varnothing = \varnothing$

$f(\{1\})$ = false, since $\{1,2\} \cap \{1\} = \{1\} \neq \varnothing$

$f(\{2,4\})$ = false, since $\{1,2\} \cap \{2,4\} = \{2\} \neq \varnothing$

* The truth set T_f is $\{\varnothing, \{3\}, \{4\}, \{3,4\}\}$.

* The same proposition f: "$\{1,2\} \cap A$" taken as a proposition over the universe $U = \mathcal{P}(\{1,2,3\})$ has truth set $\{\varnothing, \{3\}\}$.

Recall Def from Sec 3.3

* A tautology is a proposition which is true in all cases.

* A contradiction is a proposition that is false in all cases.

Def A proposition p over U is called a **tautology** if its truth set is U,

i.e. if $p(x)$ = true for all $x \in U$.

A proposition p over U is called a **contradiction** if its truth set is $\emptyset$,

i.e. if $p(x)$ = false for all $x \in U$.

Ex * The proposition "$(x-1)(x+1) = x^2 - 1$" over $U = \mathbb{Z}$ is a tautology because it is true for all $x \in \mathbb{Z}$

* The proposition "$x^2 = -1$" over $U = \mathbb{Z}$ is a contradiction because it's false for all $x \in \mathbb{Z}$.

* The proposition "$x^2 = 9$" over $U = \mathbb{Z}$ is neither a tautology nor a contradiction, since its truth set is $\{-3, 3\}$ which is neither the entire U nor $\emptyset$.

Def Suppose p and q are propositions over a universe U.

* We say p and q are <u>equivalent (over U)</u>, denoted $p \Longleftrightarrow q$, if $p \leftrightarrow q$ is a tautology, i.e., if $T_p = T_q$.

* We say <u>p implies q (over U)</u>, denoted $p \Rightarrow q$, if $p \rightarrow q$ is a tautology, i.e. if $T_p \subseteq T_q$.

Ex * "$x^2 = 9$" and "x is 3" are equivalent propositions over $U = \mathbb{N}$ — natural numbers

* "$A \cap \{4\} = \emptyset$" and "$4 \notin A$" are equivalent propositions over $\mathcal{P}(\mathbb{N})$.

Ex
- Let p be "$x \leq 4$" and q be "$x \leq 8$" over $U = \mathbb{N}$.
- Then $T_p = \{0, 1, 2, 3, 4\}$, $T_q = \{0, 1, \ldots, 8\}$.
- So $T_p \subseteq T_q$. Hence we can say: "$x \leq 4$" implies "$x \leq 8$" over $\mathbb{N}$.
- But "$x \leq 8$" does not imply $x \leq 4$ over $\mathbb{N}$.
- Over $U = \{\text{negative integers}\}$, "$x \leq 4$" $\Longleftrightarrow$ "$x \leq 8$".

Ex Let $p(x)$ be "x is positive" and $q(x)$ be "x has a square root" where $U = \mathbb{R}$. Then $T_p = (0, \infty)$, $T_q = [0, \infty)$. So $p \Rightarrow q$, but q does not imply p.

I.e. "Every positive real number has a square root" is true.
"Every real number with a square root is positive" is false.

In-class exercises ↓

(1) Over the universe $U = \mathbb{P}$, define

$p(n)$: n is prime and $n \leq 27$

$q(n)$: n is a power of 3, i.e $n = 3^k$ for $k \in \mathbb{Z}$

$r(n)$: n is a divisor of 27.

(i) Compute the truth sets of these propositions.

$T_p =$

$T_q = \{1,$

$T_r =$

(ii) Circle all correct implications.

Explain to your groupmate.

$p \Rightarrow q$ $\quad$ $q \Rightarrow r$ $\quad$ $p \Rightarrow r$

$q \Rightarrow p$ $\quad$ $r \Rightarrow q$ $\quad$ $r \Rightarrow p$

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(2) Over the universe $U = \{1, 2, \ldots, 6\}$, define

$a(x)$: x is odd

$b(x)$: $(x-1)(x-3)(x-5) = 0$

$c(x)$: $1 \leq x \leq 5$

(i) Compute the truth sets of these propositions.

$T_a =$ $\quad$ $T_b =$ $\quad$ $T_c =$

(ii) Circle all correct equivalences. Explain.

$a \Leftrightarrow b$ $\quad$ $b \Leftrightarrow c$ $\quad$ $a \Leftrightarrow c$

(1) Over the universe $U = \mathbb{P}$, define (Answers)

$p(n)$: n is prime and $n \leq 27$

$q(n)$: n is a power of 3, i.e $n = 3^k$ for $k \in \mathbb{Z}$

$r(n)$: n is a divisor of 27.

(i) Compute the truth sets of these propositions.

$T_p = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$

$T_q = \{1, 3, 9, 27, 81, \ldots\}$

$T_r = \{1, 3, 9, 27\}$

(ii) Circle all correct implications.

Explain to your groupmate.

$p \Rightarrow q$ $\quad$ $q \Rightarrow r$ $\quad$ $p \Rightarrow r$

$q \Rightarrow p$ $\quad$ $r \Rightarrow q$ $\quad$ $r \Rightarrow p$

because $T_r \subseteq T_q$

(2) Over the universe $U = \{1, 2, \ldots, 6\}$, define

$a(x)$: x is odd

$b(x)$: $(x-1)(x-3)(x-5) = 0$

$c(x)$: $1 \leq x \leq 5$

(i) Compute the truth sets of these propositions.

$T_a = \{1, 3, 5\}$ $\quad$ $T_b = \{1, 3, 5\}$ $\quad$ $T_c = \{1, 2, 3, 4, 5\}$

(ii) Circle all correct equivalences. Explain.

$a \Leftrightarrow b$ $\quad$ $b \Leftrightarrow c$ $\quad$ $a \Leftrightarrow c$

because $T_a = T_b$