Sec 3.5 Mathematical systems & proofs

Next quiz: Sec 3.2, 3.3, 3.4, HW due tonight & next Tue

Sec 3.4 tables of common implications & equivalences

Table 3.4.3. Basic Logical Laws - Equivalences

Commutative Laws	
$p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$
Associative Laws	
$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
Distributive Laws	
$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
Identity Laws	
$p \vee 0 \Leftrightarrow p$	$p \wedge 1 \Leftrightarrow p$
Negation Laws	
$p \wedge \neg p \Leftrightarrow 0$	$p \vee \neg p \Leftrightarrow 1$
Idempotent Laws	
$p \vee p \Leftrightarrow p$	$p \wedge p \Leftrightarrow p$
Null Laws	
$p \wedge 0 \Leftrightarrow 0$	$p \vee 1 \Leftrightarrow 1$
Absorption Laws	
$p \wedge (p \vee q) \Leftrightarrow p$	$p \vee (p \wedge q) \Leftrightarrow p$
DeMorgan's Laws	
$\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$	$\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$
Involution Law	
$\neg(\neg p) \Leftrightarrow p$	

Table 3.4.4. Basic Logical Laws - Common Implications and Equivalences

Detachment (AKA Modus Ponens)	$(p \to q) \wedge p \Rightarrow q$
Indirect Reasoning (AKA Modus Tollens)	$(p \to q) \wedge \neg q \Rightarrow \neg p$
Disjunctive Addition	$p \Rightarrow (p \vee q)$
Conjunctive Simplification	$(p \wedge q) \Rightarrow p$ and $(p \wedge q) \Rightarrow q$
Disjunctive Simplification	$(p \vee q) \wedge \neg p \Rightarrow q$ and $(p \vee q) \wedge \neg q \Rightarrow p$
Chain Rule	$(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$
Conditional Equivalence	$p \to q \Leftrightarrow \neg p \vee q$
Biconditional Equivalences	$(p \leftrightarrow q) \Leftrightarrow (p \to q) \wedge (q \to p) \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$
Contrapositive	$(p \to q) \Leftrightarrow (\neg q \to \neg p)$

Sec 3.5 Mathematical systems & proofs

Part I: Direct proof

Example 1

Here's a theorem: $P \vee Q,\ Q \to S,\ P \to R \Rightarrow S \vee R$

The premise

what we want to conclude

Here's a possible direct 3-column proof:

Step	Proposition	Justification
1.	$P \vee Q$	Premise
2.	$\neg P \to Q$	(1), conditional equivalence $\square \to \bigcirc \Leftrightarrow \neg\square \vee \bigcirc$ $\neg\triangle \to \bigcirc \Leftrightarrow \triangle \vee \bigcirc$
3.	$Q \to S$	Premise
4.	$\neg P \to S$	(2), (3) Chain Rule
5.	$\neg S \to P$	(4), contrapositive
6.	$P \to R$	Premise
7.	$\neg S \to R$	(5), (6), Chain Rule
8.	$S \vee R$	(7), conditional equivalence

□

Note: * Each step must be either a premise or a proposition that is implied from previous steps using any valid equivalence or implication (we'll only use Sec 3.4 tables)

* For a direct proof, the last step must be the conclusion of the theorem.

* The "justification colum" is to make the proof more readable, like the comments in a computer program.

Example 2

Here's a possible direct, 3-column proof of $\neg P \vee Q,\ S \vee P,\ \neg Q \Rightarrow S$:

Step	Proposition	Justification
1.	$\neg P \vee Q$	Premise
2.	$\neg Q$	Premise
3.	$\neg P$	(1), (2) Disjunctive simplification ($\square \vee \triangle$) and $\neg\triangle \Rightarrow \square$
4.	$S \vee P$	Premise
5.	S	(3), (4) Disjunctive simplification

Example 3 (Additional example)

We'll write a direct, 3-column proof of

$$P \to (Q \to S),\ \neg R \vee P,\ Q \Rightarrow \underbrace{(R \to S)}_{\text{Conclusion}}$$

Because the conclusion is a conditional proposition,

$$\underbrace{R}_{\text{Condition}} \to \overbrace{S}^{\text{Consequence}},$$

we can include the condition of the conclusion to the premise and then prove the consequence of the conclusion:

$$P \to (Q \to S),\ \neg R \vee P,\ Q,\ R \Rightarrow S$$

Step	Proposition	Justification
1.	$\neg R \vee P$	Premise
2.	R	Added Premise
3.	P	(1), (2), Disjunctive simplification ($\triangle \vee \square$) and $\neg \square \Rightarrow \triangle$
4.	$P \to (Q \to S)$	Premise
5.	$Q \to S$	(3), (4), Detachment ($\clubsuit \to \heartsuit$) and $\clubsuit \Rightarrow \heartsuit$
6.	Q	Premise
7	S	(5), (6), Detachment □

Part II: Indirect proof

↳ often called "proof by contradiction"

Rule To give an indirect proof of Premise $\Rightarrow$ Conclusion,

we prove that Premise, $\neg$ Conclusion $\Rightarrow$ a contradiction

Negated conclusion

Example 4 (Extra) Here's an indirect proof of the theorem:

$P \vee Q,\ Q \to S,\ P \to R \Rightarrow S \vee R$

(from our first example)

Step	Proposition	Justification
1.	$\neg(S \vee R)$	Negated conclusion
2.	$\neg S \wedge \neg R$	(1), DeMorgan's Law $\neg(\square \vee \triangle) \Leftrightarrow (\neg\square) \wedge (\neg\triangle)$
3.	$\neg S$	(2), Conjunctive simplification $\clubsuit \wedge \bigstar \Leftrightarrow \clubsuit$
4.	$Q \to S$	Premise
5.	$\neg Q$	(3), (4), Indirect Reasoning $(A \to B)$ and $\neg B \Rightarrow \neg A$
6.	$\neg R$	(2), Conjunctive simplification
7.	$P \to R$	Premise
8.	$\neg P$	(6), (7), Indirect Reasoning
9.	$(\neg P) \wedge (\neg Q)$	(5) and (8)
10.	$\neg(P \vee Q)$	(9), DeMorgan's Law
11.	$P \vee Q$	Premise
12.	0	(10) and (11) is a contradiction $\square$

Note: For an indirect proof, the last step must be a contradiction.

Example 5

Indirect proof of $a \to b, \neg(b \vee c) \Rightarrow \neg a$

1.	a	Negation of the conclusion
2.	$a \to b$	Premise
3.	b	(1), (2), detachment
4.	$b \vee c$	(3), disjunctive addition
5.	$\neg(b \vee c)$	Premise
6.	0	(4), (5) $\square$

In-class exercises ⇩

(1) Justify the steps in the following proof of

$a \vee b,\ c \wedge d,\ a \to \neg c \Rightarrow b$

Step	Proposition	Justification
(1)	$c \wedge d$	
(2)	c	
(3)	$a \to \neg c$	
(4)	$\neg(\neg c)$	
(5)	$\neg a$	
(6)	$a \vee b$	
(7)	b	

Cheat sheet ↴

Detachment (AKA Modus Ponens)	$(p \to q) \wedge p \Rightarrow q$
Indirect Reasoning (AKA Modus Tollens)	$(p \to q) \wedge \neg q \Rightarrow \neg p$
Disjunctive Addition	$p \Rightarrow (p \vee q)$
Conjunctive Simplification	$(p \wedge q) \Rightarrow p$ and $(p \wedge q) \Rightarrow q$
Disjunctive Simplification	$(p \vee q) \wedge \neg p \Rightarrow q$ and $(p \vee q) \wedge \neg q \Rightarrow p$

□

(2) Consider the following argument.

"If my salad contains watermelon, then I will carry an iPhone. I will not eat candy if I don't carry an iPhone. My salad contains watermelon. Therefore, I will eat candy."

Is this argument valid?

(Justify why or why not.)

(3) Justify the steps in the following proof of $p \to q,\ q \to r,\ \neg(p \wedge r),\ p \vee r \Rightarrow r$

Step	Proposition	Justification
(1)	$\neg r$	
(2)	$p \vee r$	
(3)	p	
(4)	$p \to q$	
(5)	q	
(6)	$q \to r$	
(7)	r	
(8)	0	

Cheat sheet ↓

Detachment (AKA Modus Ponens)	$(p \to q) \wedge p \Rightarrow q$
Indirect Reasoning (AKA Modus Tollens)	$(p \to q) \wedge \neg q \Rightarrow \neg p$
Disjunctive Addition	$p \Rightarrow (p \vee q)$
Conjunctive Simplification	$(p \wedge q) \Rightarrow p$ and $(p \wedge q) \Rightarrow q$
Disjunctive Simplification	$(p \vee q) \wedge \neg p \Rightarrow q$ and $(p \vee q) \wedge \neg q \Rightarrow p$

□

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(4) Consider the following argument.

"My snack contains chocolate chips and donut. If my snack contains apple then it doesn't contain chocolate chips. My snack contains apple or banana. Therefore, my snack contains banana."

Is this argument valid?

(Justify why or why not)

Answers

① $a \vee b$, $c \wedge d$, $a \to \neg c \Rightarrow b$

Step	Proposition	Justification
(1)	$c \wedge d$	Premise
(2)	c	Conjunctive Simplification (1)
(3)	$a \to \neg c$	Premise
(4)	$\neg(\neg c)$	Involution (2)
(5)	$\neg a$	Indirect Reasoning (3),(4)
(6)	$a \vee b$	Premise
(7)	b	Disjunctive Simplification (5),(6) □

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② The argument is not valid.

Let W : watermelon, I : iphone, C : candy

Then the argument is $W \to I$, $\neg I \to \neg C$, $W \Rightarrow C$

Let x denote the conjunction ($\wedge$) of all premises

W	I	C	$\neg I$	$\neg C$	$W \to I$	$\neg I \to \neg C$	x	$x \to C$
0	0	0	1	1	1	1	0	1
0	0	1	1	0	1	0	0	1
0	1	0	0	1	1	1	0	1
0	1	1	0	0	1	1	0	1
1	0	0	1	1	0	1	0	1
1	0	1	1	0	0	0	0	1
1	1	0	0	1	1	1	1	0
1	1	1	0	0	1	1	1	1

Answers

③ $p \to q$, $q \to r$, $\neg(p \wedge r)$, $p \vee r \Rightarrow r$

Step	Proposition	Justification
(1)	$\neg r$	Negated conclusion
(2)	$p \vee r$	Premise
(3)	p	(1), (2) Disjunctive simplification: $(\triangle \vee \square)$ and $\neg\square \Rightarrow \triangle$
(4)	$p \to q$	Premise
(5)	q	Detachment (3), (4)
(6)	$q \to r$	Premise
(7)	r	Detachment (5), (6)
(8)	0	(1), (7) is a contradiction □

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④ The argument is valid.

Let a: apple, b: banana, c: chocolate chips, d: donut.

So the argument is $c \wedge d$, $a \to \neg c$, $a \vee b \Rightarrow b$,

which we have proven in question ①

Step	Proposition	Justification
(1)	$c \wedge d$	Premise
(2)	c	Conjunctive Simplification (1)
(3)	$a \to \neg c$	Premise
(4)	$\neg(\neg c)$	Involution (2)
(5)	$\neg a$	Indirect Reasoning (3),(4)
(6)	$a \vee b$	Premise
(7)	b	Disjunctive Simplification (5),(6) □