

Sec 3.3 Equivalence and implication

Part I: Tautologies & contradictions

Consider two propositions generated by P and Q

$$x: \neg(P \wedge Q) \quad \text{and} \quad y: \neg P \vee \neg Q$$

They are different, but they have the same meaning.

Ex:

Let P : "I've gone to Middlesex Community College (MCC),

let Q : "I've gone to UMass Lowell (UML)"

Then $\neg(P \wedge Q)$ translates to "I haven't gone to both MCC and UML"

and $\neg P \vee \neg Q$ translates to "I haven't gone to MCC or I haven't gone to UML"

To see this for sure, write the truth tables

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

In all cases, $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ have the same truth value.

When we apply the biconditional operator $\leftrightarrow$ to them, the result is a value of true in all cases:

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$
0	0	1	1	1
0	1	1	1	1
1	0	1	1	1
1	1	0	0	1

The proposition " $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$ " is an example of a tautology.

Def A tautology is a proposition which is true in all cases.

Notation: 1 is used to symbolize a tautology

Other examples of tautologies

- $P \vee \neg P$ is a tautology

Truth table of $P \vee \neg P$:

P	$\neg P$	$P \vee \neg P$
0	1	1
1	0	1

- $(P \wedge Q) \rightarrow P$ is a tautology

Truth table of $(P \wedge Q) \rightarrow P$:

p	q	$p \wedge q$	$p \wedge q \rightarrow p$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	1

- $Q \rightarrow (P \vee Q)$ is also a tautology

Construct the truth table.

- $P \rightarrow Q$ is not a tautology.

In some cases, this proposition is not true:

P	Q	$P \rightarrow Q$
0	0	1
0	1	1
1	0	0 
1	1	1

Ex "I've been to UML and I haven't been to UML"
is an example of a contradiction.

In symbolic notation: $P \wedge \neg P$

Def A contradiction is a proposition that
is false in all cases.

Notation: 0 is used to symbolize a contradiction.

Other examples of contradictions

- $(P \vee Q) \wedge (\neg P) \wedge (\neg Q)$ is a contradiction

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$(P \vee Q) \wedge (\neg P) \wedge (\neg Q)$
0	0	0	1	1	0
0	1	1	1	0	0
1	0	1	0	1	0
1	1	1	0	0	0

Part II: Equivalence

- Recall from earlier that the proposition " $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$ " is an example of a tautology

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$
0	0	1	1	1
0	1	1	1	1
1	0	1	1	1
1	1	0	0	1

We say $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are equivalent,

and write $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

ended here Week 6 Tue

Def Two propositions x and y (generated by the same set S of propositions) are said to be equivalent, denoted $x \Leftrightarrow y$

if and only if the proposition $x \leftrightarrow y$ is a tautology.

Remark Equivalence is to logic what equality is to algebra. Just as there are many ways of writing an algebraic expression (ex: $1+2 = 0+3 = 6/2$), the same logical meaning can be expressed in many different ways.

Ex $(P \wedge Q) \vee (\neg P \wedge Q) \Leftrightarrow Q$

Truth table:

P	Q	$P \wedge Q$	$\neg P$	$\neg P \wedge Q$	$(P \wedge Q) \vee (\neg P \wedge Q)$
0	0	0	1	0	0
0	1	0	1	1	1
1	0	0	0	0	0
1	1	1	0	0	1

Same truth values for all cases

Exercise :

Write the truth tables of

$$P \rightarrow Q, \quad \neg Q \rightarrow \neg P, \quad P \vee Q, \quad \neg P \vee Q$$

Are any two of these equivalent to one another?

Ans $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P \Leftrightarrow \neg P \vee Q$
"contrapositive" "Conditional equivalence"
in Table 3.4.4

Fact All tautologies are equivalent to one another.

Ex $P \vee \neg P \Leftrightarrow (P \wedge Q) \rightarrow P \Leftrightarrow q \rightarrow (P \vee Q) \Leftrightarrow 1$

Fact All contradictions are equivalent to one another.

Ex $P \wedge \neg P \Leftrightarrow (P \vee Q) \wedge (\neg P) \wedge (\neg Q) \Leftrightarrow 0$

Q Can we write $P \Leftrightarrow Q$?

Ans No, because $P \Leftrightarrow Q$ is not a tautology.

P	Q	$P \Leftrightarrow Q$
0	0	0
0	1	1
1	0	0
1	1	1

↑
not all true

Part III: Implication

- Recall earlier that $(P \wedge Q) \rightarrow P$ is a tautology

Truth table of $(P \wedge Q) \rightarrow P$:

p	q	$p \wedge q$	$p \wedge q \rightarrow p$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	1

We say that " $(P \wedge Q)$ implies P " or " P is implied by $P \wedge Q$ "

and write " $(P \wedge Q) \Rightarrow P$ "

Def Let x and y be two propositions (generated by the same set S of propositions). We say x implies y

if the proposition $x \rightarrow y$ is a tautology. Write

$$x \Rightarrow y$$

to indicate this implication.

Q Can we write $P \Rightarrow Q$?

Ans No, because $P \rightarrow Q$ is not a tautology.

P	Q	$P \rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

not all true

Group exercise:

- (a) Construct the truth table for $x: P \wedge Q \rightarrow \neg R$
and for $y: P \wedge R \rightarrow \neg Q$

(Half the group works out the column for x
& half does the column for y)

- (b) From this table, determine whether $x \Leftrightarrow y$ yes

p	q	r	$\neg q$	$\neg r$	$p \wedge q$	$p \wedge r$	$p \wedge q \rightarrow \neg r$	$p \wedge r \rightarrow \neg q$
0	0	0	1	1	0	0	1	1
0	0	1	1	0	0	0	1	1
0	1	0	0	1	0	0	1	1
0	1	1	0	0	0	0	1	1
1	0	0	1	1	0	0	1	1
1	0	1	1	0	0	1	1	1
1	1	0	0	1	1	0	1	1
1	1	1	0	0	1	1	0	0

Group exercise: (From Exercise #2 & HW 7)

(a) Construct the truth table for $x = (p \wedge \neg q) \vee (r \wedge p)$

(b) From this table, determine whether

$$p \wedge q \wedge r \Rightarrow x \quad \text{yes}$$

(c) From this table, determine whether

$$p \wedge \neg q \Rightarrow x \quad \text{yes}$$

(d) From this table, determine whether

$$x \Rightarrow p \vee q \quad \text{yes}$$

p	q	r	$\neg q$	$p \wedge \neg q$	$r \wedge p$	$(p \wedge \neg q) \vee (r \wedge p)$
0	0	0	1	0	0	0
0	0	1	1	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	0	0
1	0	0	1	1	0	1
1	0	1	1	1	1	1
1	1	0	0	0	0	0
1	1	1	0	0	1	1