* HW 4 due tonight
* Thurs Quiz : Sec 1.5, 2.1, 2.2
(Practice w/ HW incl. optional)
* Exam 1 (Thurs) : Ch 1 and 2

Review of Sec 2.1, 2.2

Ex How many different strings can be produced by rearranging the letters in the word LOWELL?

Sol 1 _ _ _ _ _ _

There are six places to put letter O,
there are five places to put letter W,
" four " E.

(In the blank places, we put the letter L)

There are $6 \cdot 5 \cdot 4 = 120$ different strings that could be produced

Sol 2 Number of permutations of 6 objects is $6!$
But we don't distinguish between the first, 2nd, 3rd L's.
There are $3!$ ways to rearrange these three L's.
So there are $\frac{6!}{3!} = 6 \cdot 5 \cdot 4 = \boxed{120 \text{ strings}}$

Ex How many different strings can be produced by rearranging the letters in MASSACHUSETTS?

Ans $$\boxed{\frac{13!}{\underbrace{2!}_{\text{Two A's}}\ \underbrace{4!}_{\text{Four S's}}\ \underbrace{2!}_{\text{Two T's}}} \text{ strings}}$$

Sec 2.4 Combinations & the binomial Thm

Combinations

Ex (Counting permutations)

I have exactly five scoops of ice cream:

Apple, Banana, Chocolate, Durian, Espresso

How many different ways are there to form triple-scoop ice cream cones?

$\leftarrow p_3 = 3$

$\leftarrow p_2 = 4$

$\leftarrow p_1 = 5$

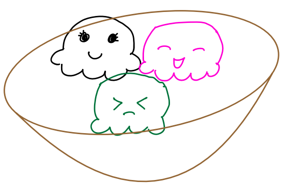
Answer: $5 \cdot 4 \cdot 3 = 60$

or $P(5,3) = \dfrac{5!}{(5-3)!} = \dfrac{5!}{2!}$ ways

Note: The order matters!

Ex (Counting with no order)

How many ways are there to form triple-scoop ice cream bowls?



Note: the order doesn't matter!

This is equivalent to asking: How many 3-element subsets of the set $\{A,B,C,D,E\}$ are there?

Let's first list all such subsets:

$\{A,B,C\}$, $\{A,B,D\}$, $\{A,B,E\}$, $\{A,C,D\}$, $\{A,C,E\}$, $\{A,D,E\}$,
$\{B,C,D\}$, $\{B,C,E\}$, $\{B,D,E\}$, $\{C,D,E\}$

Answer: 10 ways

The notation for choosing 3 elements from 5 is

$$\binom{5}{3} \quad \text{or} \quad C(5,3)$$

Read: "5 choose 3"

Read: "number of combinations for 5 objects taken 3 at a time"

In general, we write ...

Def (Binomial Coefficient)

Let n and k be non negative integers.

The binomial coefficient $\binom{n}{k}$ represents the number of combinations of n objects taken k at a time.

Read $\binom{n}{k}$ as "n choose k".

Let's investigate the relationship between counting permutations & combinations, to try to derive a formula for $\binom{n}{k}$.

Let's consider $\binom{4}{3}$ = # of 3-element subsets of $\{A, B, C, D\}$.

All such subsets:	Their permutations
$\{A, B, C\}$	ABC, BAC, ACB, BCA, CAB, CBA
$\{A, B, D\}$	ABD, ... , DBA
$\{A, C, D\}$	ACD, ... , DCA
$\{B, C, D\}$	BCD, ... , DCB

Thus, $\binom{4}{3} = \frac{P(4,3)}{3!} = \frac{4!}{(4-3)!} \frac{1}{3!} = 4$

Thm (Binomial Coefficient Formula)

Let n and k be nonnegative integers with $0 \le k \le n$. Then the number $\binom{n}{k}$ of k-element subsets of an n-element set is equal to
$$\frac{n!}{(n-k)!\,k!}$$

Proof Given a k-element subset, there are $k!$ ways of ordering the elements, so
$$\binom{n}{k} = \frac{P(n,k)}{k!} = \underset{P(n,k)}{\frac{n!}{(n-k)!}} \frac{1}{k!}$$
$\square$

Alternative proof

To construct a permutation of k objects from a set of n elements:

Step 1: Choose a k-subset of the n objects $\binom{n}{k}$

Step 2: Choose one of the $k!$ permutations of those k objects.

Thus, $P(n,k) = \underset{\text{Step 1}}{\binom{n}{k}}\ \underset{\text{Step 2}}{k!}$

$\square$

Alternative solutions using combination

Ex How many different strings can be produced by rearranging the letters in the word LOWELL?

Sol 3 $\_\ \_\ \_\ \_\ \_\ \_$

Choose 3 of the 6 positions to hold the three L's, then choose 1 of the remaining 3 positions to hold an O, then choose 1 of the remaining 2 positions to hold a W, then choose 1 of the remaining 1 position to hold an E.

There are $\binom{6}{3}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \frac{6!}{3!3!}\ \frac{3!}{1!2!}\ \frac{2!}{1!1!}\ 1$

$$= 20 \cdot 3 \cdot 2$$

$$= 120$$

different strings that could be produced

<u>Ex</u> How many different strings can be produced by rearranging the letters in MASSACHUSETTS ?

<u>Alternative solution using combination</u>

Choose 1 of the 13 positions to hold an M,
then choose 2 of the remaining 12 positions to hold A's,
then choose 4 of the remaining 10 positions to hold S's,
then choose 1 of the remaining 6 positions to hold a C,
then choose 1 of the remaining 5 positions to hold an H,
then choose 1 of the remaining 4 positions to hold a U,
then choose 1 of the remaining 3 positions to hold an E,
then choose 2 of the remaining 2 positions to hold T's.

There are $\binom{13}{1}\binom{12}{2}\binom{10}{4}\binom{6}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{2}$

$$= 13 \; \frac{12!}{2!\cancel{10!}} \; \frac{\cancel{10!}}{4!6!} \; 6 \cdot 5 \cdot 4 \cdot 3 \cdot 1$$

$$= 13 \; \frac{12!}{2} \; \frac{1}{4! \; \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot 2 \cdot 1} \; \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot 1$$

$$= \frac{13!}{2! \; 4! \; 2!}$$ possible strings

Ex (Flipping coins)

We have one coin, and we toss it five times.

In how many ways can we get exactly three heads?

Ans Think of the five flips as a set

$$\{\text{flip } 1, \ \text{flip } 2, \ \text{flip } 3, \ \text{flip } 4, \ \text{flip } 5\}$$

There are $\binom{5}{3} = \frac{5!}{3!2!} = \frac{5 \cdot 4}{2} = 10$ ways to get exactly 3 heads:

$$\{1, 2, 3\} \longleftrightarrow H\ H\ H\ T\ T$$
$$\{1, 2, 4\} \longleftrightarrow H\ H\ T\ H\ T$$
$$\{1, 2, 5\} \longleftrightarrow H\ H\ T\ T\ H$$
$$\vdots \qquad\qquad \vdots$$
$$\{3, 4, 5\} \longleftrightarrow T\ T\ H\ H\ H$$

Ex Counting five flips, using combinations (Sec 2.4) vs using rule of products (Sec 2.1)

What is the total number of possible outcomes of tossing my (one) coin five times?

1 We can partition the possibilities into these pairwise disjoint events:

5 heads, 4 heads, 3 heads, 2 heads, 1 head, 0 head

$\{HHHHH\}$ $\{HTTTT, THTTT, \ldots\}$ $\{\quad\}$ $\{\quad\}$ $\{\quad\}$ $\{TTTTT\}$

For example, there are $\binom{5}{3}$ events in which 3 heads appear.
By the law of addition, we have

$$\binom{5}{5} + \binom{5}{4} + \binom{5}{3} + \binom{5}{2} + \binom{5}{1} + \binom{5}{0}$$

possible outcomes.

2 Since there are two possible outcomes for each of the five tosses, by the rule of products, we have $2^5 = 32$ outcomes.

The previous example suggests that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

— Stop —

The Binomial Theorem

The binomial theorem is a formula for expanding $(x+y)^n$, where n is a nonnegative integer.

$$(x+y)^0 = 1$$

$$x+y = x+y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$\vdots$$

Record just the coefficients in an array

row 0						1						
row 1					1		1					
row 2				1		2		1				
row 3			1		3		3		1			
row 4		1		4		6		4		1		
row 5	1		5		10		10		5		1	

$$\vdots$$

(called Pascal's triangle, Yang Hui triangle, the triangle of Omar Khayyam, the triangle of Tartaglia, and the Staircase of Mount Meru)

Fact

For $n = 0, 1, \ldots$ and $k = 0, 1, 2, \ldots, n$, the k-th entry in row # n is the coefficient of $x^{n-k}y^k$ in the expansion of $(x+y)^n$

The coefficient for x^3y^2 in the expansion of $(x+y)^5$ is $\binom{5}{2}$ = # of ways to toss a coin (w/ sides labeled x and y) five times and get exactly 2 y's.

$$(x+y)(x+y)(x+y)(x+y)(x+y) = x^5 + \underbrace{5x^4y}_{\text{five options to choose just one } y} + 10x^3y^2 + \dots$$

<u>Binomial Thm</u> If $n \geqslant 0$ and x, y are numbers, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

<u>Ex</u> Find the third term in the expansion of

$$(x-y)^4 = (x+(-y))^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3(-y) + \binom{4}{2}x^2(-y)^2 + \dots$$

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2} = 6$$

So the third term is $\boxed{6x^2y^2}$

<u>Ex</u> What is the coefficient of x^2y^3 in the expansion of $(2x-y)^5$?

<u>Ans</u> $(a+b)^5 = \dots + \binom{5}{3}a^2b^3 + \dots$ by Binomial Theorem

$$(2x+(-y))^5 = \dots + \binom{5}{3}(2x)^2(-y)^3 + \dots$$

$$\binom{5}{3}(2x)^2(-y)^3 = 10\,(4)\,(-1)\,x^2y^3 = -40x^2y^3$$

So the coefficient of x^2y^3 is $\boxed{-40}$

—end—