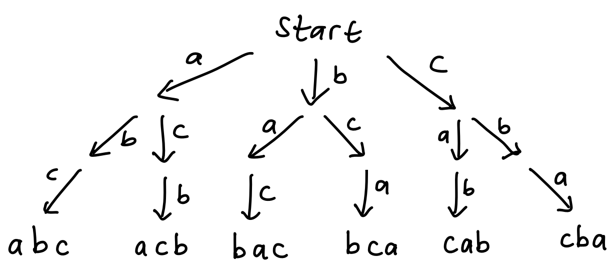
Sec 2.2 Permutations

Ex (Ordering the elements of a set)

How many different ways can we order the elements of the set $A = \{a, b, c\}$?

Sol Draw a tree diagram (optional):



three choices

two choices

one choice

By the rule of products, we have $3 \cdot 2 \cdot 1$ different ways of ordering the three letters

Each of these six orderings,

abc, acb, bac, bca, cab, cba

is called a <u>permutation</u> of the set A.

Ex Consider a basketball team with 25 playes.

- The alphabetical ordering of the players' names is one permutation of the set P of players
- Ordering of the players by age is also a permutation of P.
- Any way to list the players is a permutation of P.

There are $\underbrace{25 \cdot 24 \cdot 23 \cdot \ldots \cdot 3 \cdot 2 \cdot 1}$ permutations of the player

There are 26 digits of this number, so for short write $25!$

Def If $n \in \underbrace{\mathbb{P}}_{\text{positive integers}}$, then $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

Read "n factorial"

Define $0!$ to be 1.

The first few factorials

n	0	1	2	3	4	5	6	7
$n!$	1	1	2	6	24	120	720	5040

Ex Suppose all 25 players on the basketball team are capable of playing any position.

Q: How many ways can the coach fill the five starting positions in a game?

Ans By the rule of products, there are

$$25 \cdot 24 \cdot 23 \cdot 22 \cdot 21$$

ways of making a selection.

Any of these selections is called a "permutation of 25 players taken 5 at a time"

Def
- An ordered arrangement of k elements from a set of n elements, $0 \leq k \leq n$, where no two elements of the arrangement are the same, is called a <u>permutation of k objects taken from a set of n elements</u>
- The total number of permutations of k elements taken from a set of n elements is denoted by $P(n,k)$

Theorem (Permutation counting formula)

$$P(n,k) = \underbrace{n\cdot(n-1)\cdot(n-2)\cdots(n-(k-1))}_{k\text{ factors}} = \prod_{j=0}^{k-1}(n-j)$$

Proof

Position 1 can be filled by any of the $n-0=n$ elements

— " — 2 ———— " ———— $n-1$ elements

...

Position k can be filled by any of the $n-(k-1)$ elements

The result then follows from the rule of products. $\square$

Note $P(n,k) = n\cdot(n-1)\cdot(n-2)\cdots(n-(k-1))$

Can be written more compactly as $\dfrac{n!}{(n-k)!}$

For example, if $n=25$ and $k=5$, we have

$$\frac{n!}{(n-k)!} = \frac{25!}{20!} = \frac{25\cdot 24\cdot 23\cdot 22\cdot 21\cdot \cancel{20}\cdot\cancel{19}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{20}\cdot\cancel{19}\cdots\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

—— ended here week 3 Thurs ——

Ex (From previous class' activity)

Consider three-digit numbers that can be formed from the digits 3, 4, 5, 6 and 7 with no repetition of digits allowed.

a) How many are there?

$5 \cdot 4 \cdot 3 = 60$ numbers

Note this is $P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3$

b) How many of these are even numbers?

solution 1

Choose the last digit (either 4 or 6): $p_1 = 2$ options

— " — middle digit: $p_2 = 5-1 = 4$ options

— " — first digit: $p_3 = 4-1 = 3$ options.

There are $2 \cdot 4 \cdot 3 = 24$ even numbers

solution 2

There are $5 \cdot 4 \cdot 3 = 60$ total three-digit numbers with no repetition. Of these, $\frac{2}{5}$ end in a 4 or 6.

So the number of three-digit even numbers with no repetition is $\frac{2}{5} \, 5 \cdot 4 \cdot 3 = 24$.

Ex How many different strings can be produced by rearranging the letters in the word LOWELL?

Sol 1 _ _ _ _ _ _

There are six places to put letter O,
there are five places to put letter W,
— " — four — " — E.

(The other blank places, we put the letter L)

There are $6 \cdot 5 \cdot 4 = 120$ different strings that could be produced

Sol 2
Number of permutations of 6 objects is $6!$
But we don't distinguish between the first, 2nd, 3rd L's.
There are $3!$ ways to rearrange these three L's.
So there are $\frac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120$ strings

Ex How many different strings can be produced by rearranging the letters in MASSACHUSETTS?

Ans

$$\frac{13!}{\underbrace{2!}_{\text{Two A's}}\ \underbrace{4!}_{\text{Four S's}}\ \underbrace{2!}_{\text{Two T's}}}$$