→ Check your score for HW 3.
You can resubmit until you get 100 %

→ Quiz 2 on Thurs (Sec 1.3 and 1.4)

Sec 2.1 Basic counting techniques – the rule of products

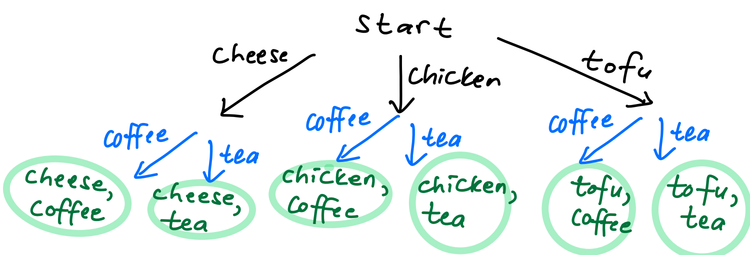
Ex (Lunch)

A café serves three different sandwiches (cheese, chicken, tofu) and two different beverages (coffee and tea).

Q: How many different lunches can a person order?

Method 1: List all possibilities, then count them

Draw a tree diagram:



There are $\boxed{6 \text{ possibilities}}$

Note: 3 arrows in the first step,
2 arrows each in the second step.

Method 2: Realize that there are 3 different choices for food and 2 different choices for beverages, so there are $3 \cdot 2 = 6$ different lunches that can be ordered.

Ex (Elements in a Cartesian product)

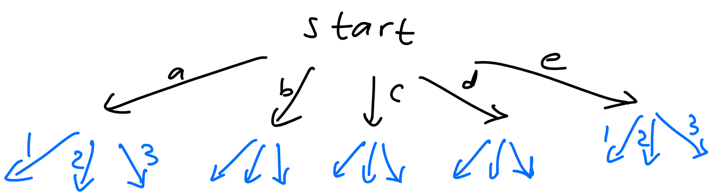
Let $S = \{a, b, c, d, e\}$, $T = \{1, 2, 3\}$.

Then $S \times T \underset{\text{def}}{=} \{(s,t) \mid s \in S,\ t \in T\}$

$= \{(a,1), (a,2), (a,3), \ldots, (e,2), (e,3)\}$

Q: How many elements are in $S \times T$?

Draw the tree diagram:



Observe again 5 arrows in the first step.

For each of the 5 arrows, we have 3 arrows.

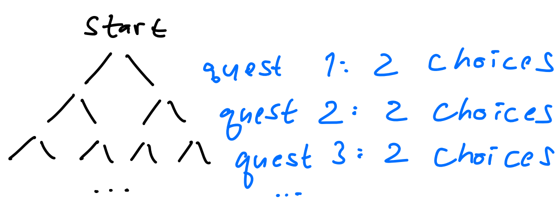
$S \times T$ has $5 \cdot 3 = \boxed{15}$ elements.

Ex (True-False Questionnaire)

A person is completing a true-false questionnaire consisting of ten questions.

Q: How many different ways are there to answer the questionnaire?

Imagine the tree diagram (but don't draw it!)



There are $2 \cdot 2 \cdot 2 \cdots 2 = 2^{10} = \boxed{1024}$ ways to answer the questionnaire

The Rule of products

Thm (The rule of products)

- Suppose two operations must be performed.
- Suppose the first operation can always be performed p_1 different ways, and the second operation can always be performed p_2 different ways.
- Then there are $p_1 p_2$ ways that the two operations can be performed

Thm In general: If n operations must be performed, & the number of options for each operation is $p_1, p_2, \dots, p_n$ respectively, with each p_i independent of previous choices, then the n operations can be performed $p_1 \cdot p_2 \cdot \dots \cdot p_n$ different ways.

Warning: The order that the operations are performed matters.

Ex How many triple-scoop ice cream cones can you form if there are 10 flavors to choose from? (Flavors can be repeated)

$\leftarrow p_3 = 10$

$\leftarrow p_2 = 10$

$\leftarrow p_1 = 10$

Answer: $10 \cdot 10 \cdot 10 = 1000$ possible 3-scoop ice cream cones

Ex How many triple-scoop ice cream cones can you form if there are 10 flavors to choose from and flavors cannot be repeated?

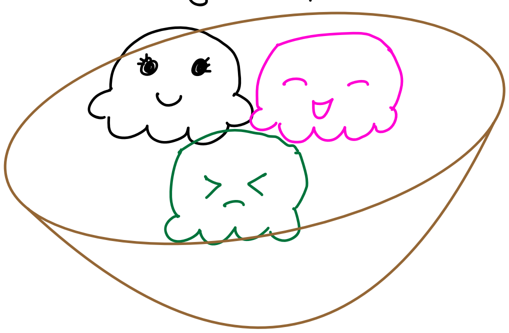
$\leftarrow p_3 = 8$

$\leftarrow p_2 = 9$

$\leftarrow p_1 = 10$

Answer: $10 \cdot 9 \cdot 8 = 720$ possible 3-scoop ice cream cones

Ex How many triple-scoop ice cream bowls can you form?



Because the ice cream gets mixed in the bowl, the order doesn't matter!

We will learn about this situation (where the order doesn't matter) soon, in Sec 2.4.

Thm (Power set cardinality theorem)

If A is a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$

(Recall: $\mathcal{P}(A)$ is the power set of A, i.e $\mathcal{P}(A) = \{S \mid S \text{ is a subset of } A\}$)

Proof Let n denote $|A|$. Let $S \in \mathcal{P}(A)$.

For each element $x \in A$ there are 2 choices: $x \in S$ or $x \notin S$.

By the rule of products, we have

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ factors}} = 2^n$$

different subsets of A.

— begin group work —

Ex (Exercise # 10)

A typical Massachusetts license plate consists of three digits followed by three letters. The first digit is never zero. How many different plates of this type could be made?

Ans

$$\overset{P_1}{9} \cdot \underbrace{\overset{P_2}{10} \cdot \overset{P_3}{10}}_{} \cdot \underbrace{\overset{P_4}{26} \cdot \overset{P_5}{26} \cdot \overset{P_6}{26}}_{} = 9 \cdot 10^2 \cdot 26^3 \approx 15 \text{ million}$$

first digit could be 1, 2, …, or 9

2nd & third digits could be 0, 1, …, 9

three letters can be a, b, c, …, z

<u>Ex</u> How many proper subsets of $A = \{1,2,3,4,5\}$ are there?

<u>Ans</u> Recall a subset S of a set T is called <u>proper</u> if $S \neq T$.

The set of all proper subsets of A is

$$\mathcal{P}(A) - \{A\},$$

so the number of proper subsets of A is $2^{|A|} - |\{A\}|$

$$= 2^5 - 1$$

$$= 32 - 1$$

$$= 31$$

<u>Ex</u> How many subsets of $A = \{1,2,3,4,5\}$ contain 1 and 2?

<u>Ans</u> The same number as $|\mathcal{P}(\{3,4,5\})| = 2^3 = 8$

<u>Ex</u> How many subsets of $A = \{1,2,3,4,5\}$ does not contain 3?

<u>Ans</u> The same number as $|\mathcal{P}(\{1,2,4,5\})| = 2^4 = 16$

Warning: The number P_2 must not depend on the option that is chosen in the first operation, i.e. P_2 must be independent of the first operation. Otherwise, the "rule of the product" does not apply.

Ex Consider three-digit numbers that can be formed from the digits 3, 4, 5, 6 and 7 with no repetition of digits allowed.

a) How many are there?

$5 \cdot 4 \cdot 3 = 60$ numbers

b) How many of these are even numbers?

A wrong "solution":

There are 5 ways to choose the first digit, and then 4 ways to choose the second digit.

The number of ways to choose the (even) third digit would be:

0 if the first two digits are both even

1 if the first two digits have one even digit and one odd digit

2 if the first two digits are both odd.

Since the number of options for the third choice depends on what the first two choices were, the Law of Products does NOT apply.

Correct solution 1

Choose the last digit (either 4 or 6): $p_1 = 2$ options

— " — middle digit: $p_2 = 5 - 1 = 4$ options

— " — first digit: $p_3 = 4 - 1 = 3$ options.

There are $2 \cdot 4 \cdot 3 = 24$ possibilities

Correct solution 2

There are $5 \cdot 4 \cdot 3 = 60$ total three-digit numbers with no repetition. Of these, $\frac{2}{5}$ end in a 4 or 6.

So the number of three-digit even numbers with no repetition is $\frac{2}{5}\, 5 \cdot 4 \cdot 3 = 24$.

— end —