Sec 1.5 Summation notation & generalizations

- A <u>binary operation</u> is taking an ordered pair a, b as input and producing one output.

<u>Ex</u> Division $\div$ is a binary operation.

- Addition is an example of an <u>associative</u> operation:

$((a_1 + a_2) + a_3) + a_4$ and $(a_1 + a_2) + (a_3 + a_4)$

are the same. No need to put parentheses.

- Multiplication is also associative:

$(a_1 \cdot a_2) \cdot a_3$ and $a_1 \cdot (a_2 \cdot a_3)$ are the same

- Examples of non-associative operations ?

- A <u>finite series</u> is a sum of finitely many numbers,

like $\quad 1 + 200 + 50 + (-\frac{1}{2}) + 0$

or $\quad a_1 + a_2 + a_3 + a_4 = \underbrace{\sum_{k=1}^{4} a_k}_{\text{summation notation}}$

i.e. a finite series is an expression of the form

$$a_1 + a_2 + \dots + a_n = \sum_{k=1}^{n} a_k$$

Here: • k is the <u>index</u> of the summation.
Other letters like $i, j, \ell, m, \dots$ are also often used

• a_k is the <u>general term</u> of the series

• $\sum_{k=i}^{n} a_k$

n – terminal index

i – initial index, can be any integer

Ex $$\sum_{k=0}^{4} b_k = b_0 + b_1 + b_2 + b_3 + b_4$$

$$\sum_{j=-2}^{1} c_j = c_{-2} + c_{-1} + c_0 + c_1$$

Note the letter used for the index must be consistent

Ex Sometimes the general terms follow a specific pattern and can be simplified:

$$\sum_{l=1}^{3} l^2 = 1^2 + 2^2 + 3^2 = 14$$

$$\sum_{i=1}^{4} (2i-1) = 1 + 3 + 5 + 7 = 16$$

The summation notation can be generalized to other associative operations.

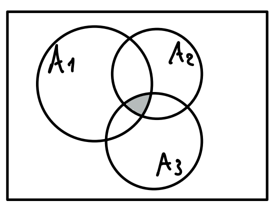
Multiplication : $\prod_{k=1}^{n} a_k = a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdots a_n$

Ex $$\prod_{k=1}^{3} k^2 = 1^2 \cdot 2^2 \cdot 3^2 = 36$$

Set intersection : $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$

Ex Draw the Venn diagram for $\bigcap_{i=1}^{3} A_i$

Sol $$\bigcap_{i=1}^{3} A_i = A_1 \cap A_2 \cap A_3$$

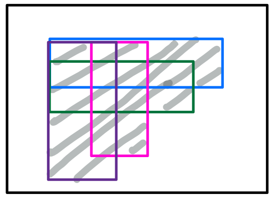


Note: $2^3 = 8$ possible regions for whether an element is in A_i or not, for $i = 1, 2, 3$.

Set union: $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cdots \cup A_n$

Ex Draw the Venn diagram for $\bigcup_{i=1}^{4} A_i$

Sol $\bigcup_{i=1}^{4} A_i = A_1 \cup A_2 \cup A_3 \cup A_4$

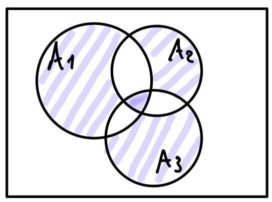


Note: $2^4 = 16$ possible regions for whether or not an element is in A_i or not, for $i = 1, 2, 3, 4$.

Symmetric difference: $\bigoplus_{i=1}^{n} A_i = A_1 \oplus A_2 \oplus \cdots \oplus A_n$

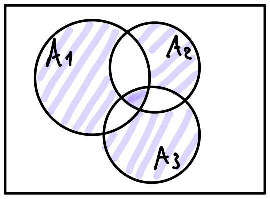
Ex Draw the Venn diagram for $\bigoplus_{i=1}^{3} A_i$

Sol $\bigoplus_{i=1}^{3} A_i = A_1 \oplus A_2 \oplus A_3$



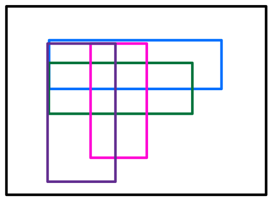
Note $\bigoplus_{i=1}^{3} A_i$ is the set of elements that belong to exactly one of the sets or that belong to all three sets.

Recall $\bigoplus_{i=1}^{3} A_i$ is the set of elements that belong to exactly one of the sets or that belong to all three sets.

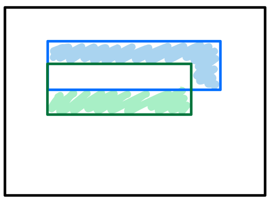


Q: What about $\bigoplus_{i=1}^{4} A_i = A_1 \oplus A_2 \oplus A_3 \oplus A_4$?

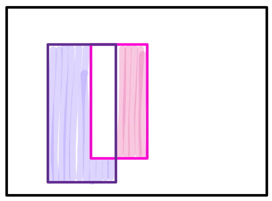
Venn diagram for 4 sets:



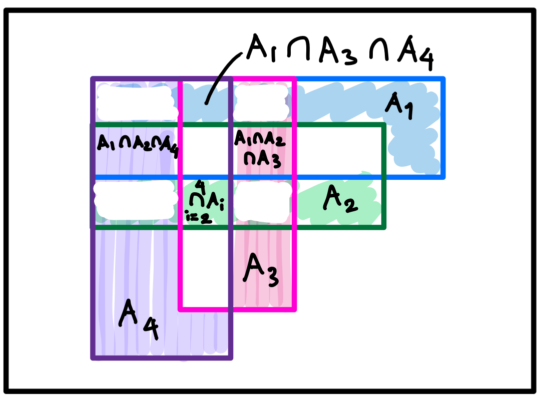
$A_1 \oplus A_2$



$A_3 \oplus A_4$



$(A_1 \oplus A_2) \oplus (A_3 \oplus A_4)$



$$(A_1 \oplus A_2) \oplus (A_3 \oplus A_4) = \left\{ x \;\middle|\; \begin{array}{l} x \text{ is in exactly 1 of the } A_i\text{'s or} \\ x \text{ is in exactly 3 of the } A_i\text{'s} \end{array} \right\}$$

<u>Ex</u> Let $A_1 = \{0, 2, 3\}$, $A_2 = \{1, 2, 3, 6\}$, and $A_3 = \{-1, 0, 3, 9\}$.

Then $\bigcap_{i=1}^{3} A_i = A_1 \cap A_2 \cap A_3 = \{3\}$

$$\bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3 = \{-1, 0, 1, 2, 3, 6, 9\}$$

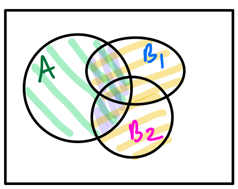
$$\bigoplus_{i=1}^{3} A_i = \{-1, 1, 3, 6, 9\}$$

Extra Examples

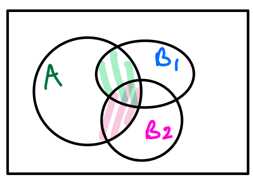
Cartesian product: $\bigtimes_{i=1}^{n} A_i = A_1 \times A_2 \times \dots \times A_n$

<u>Fact</u> $A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2)$ for sets A, B_1, B_2

Why?



LHS



RHS

In general, $A \cap (B_1 \cup B_2 \cup \dots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_n)$ for sets $A, B_1, B_2, \dots, B_n$.

More compactly, write $A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$

Exercise #5

Expand the expression $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

for $n=1$: $(x+y)^1 = \sum_{k=0}^{1} \binom{n}{k} x^{n-k} y^k = \binom{1}{0} x + \binom{1}{1} y$

for $n=2$: $(x+y)^2 = \binom{2}{0} x^2 + \binom{2}{1} xy + \binom{2}{2} y^2$

for $n=3$: $(x+y)^3 = \binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} xy^2 + \binom{3}{3} y^3$

3.

(a) Express the formula $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ without using summation notation.

(b) Verify this formula for $n = 3$.

(c) Repeat parts (a) and (b) for $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Answer.

(a) $$\frac{1}{1(1+1)} + \frac{1}{2(2+1)} + \frac{1}{3(3+1)} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

(b) $$\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4} = \frac{3}{3+1}$$

(c) $1 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{1}{4}\right) n^2(n+1)^2$ $\quad 1 + 8 + 27 = 36 = \left(\frac{1}{4}\right)(3)^2(3+1)^2$

Exercise #7

Let $A_k = \{x \in \mathbb{Q} \mid k-1 < x \leq k\}$

(1) What is $\bigcup_{i=1}^{3} A_i$?

Ans $A_1 = \{x \in \mathbb{Q} \mid 0 < x \leq 1\} = \{$ rational numbers in $(0,1]\}$

$A_2 = \{x \in \mathbb{Q} \mid 1 < x \leq 2\} = \{$ rational numbers in $(1,2]\}$

$A_3 = \{x \in \mathbb{Q} \mid 2 < x \leq 3\} = \{$ rational numbers in $(2,3]\}$

$$\bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3 = \{x \in \mathbb{Q} \mid 0 < x \leq 3\}$$

(2) What is $\bigcap_{i=1}^{3} A_i$? $\varnothing$

—end—