Sec 1.3 Cartesian products & power sets

* HW 2 due tonight
* HW 3 due next Tue
  → you can "resubmit" as many times as you want (prior to deadline)

* Quiz 1 this Thurs
  → 10 mins long, at the end of class
  → Topics: Sec 1.1, 1.2
  → No notes, no electronics

Sec 1.3 Cartesian products & power sets

Cartesian product

Def The Cartesian product of two sets A and B

is $A \times B = \{ (a,b) \mid a \in A, \ b \in B \}$,

the order matters!

i.e. $A \times B$ is the set of all possible ordered pairs

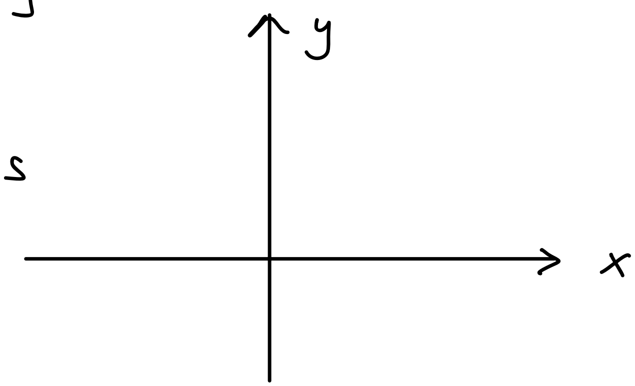
whose 1st component comes from A &

whose 2nd —— " —— B

Ex $\mathbb{R} \times \mathbb{R} = \{ \qquad \}$

or $\mathbb{R}^2$

Can be visualized as points in the xy-plane



Cartesian product

Def The Cartesian product of two sets A and B

is $A \times B = \{ (a,b) \mid a \in A, \ b \in B \}$,

Ex The elements of the product $\{1, 2, 3\} \times \{\heartsuit, \square\}$ are :

1. $(1, \heartsuit)$
2.
3.
4.
5.
6.

Note Is $\{\heartsuit, \square\} \times \{1, 2, 3\}$ equal to $\{1, 2, 3\} \times \{\heartsuit, \square\}$?

No

Cartesian product

<u>Def</u> The <u>Cartesian product</u> of two sets A and B

is $$A \times B = \{ (a,b) \mid a \in A ,\ b \in B \},$$

<u>Note</u> Suppose A, B are nonempty sets.

When are $A \times B$ and $B \times A$ equal?

When $A = B$

Cartesian product

<u>Def</u> The <u>Cartesian product</u> of two sets A and B

is $A \times B = \{ (a,b) \mid a \in A, \ b \in B \}$,

<u>Ex</u> $\mathbb{N} \times \{1,2\} = \{ (a,b) \mid a \in \mathbb{N}, \ b \in \{1,2\} \}$

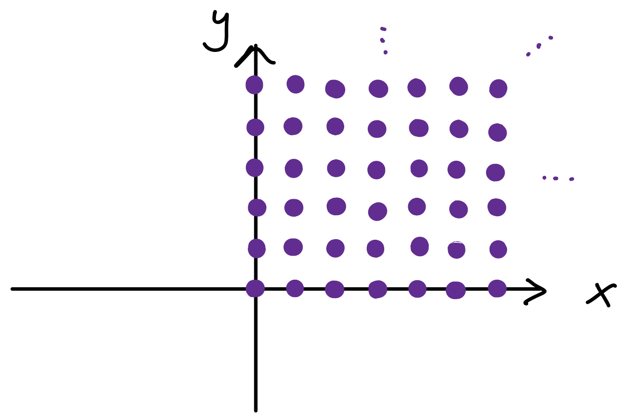
$$
\begin{aligned}
= \{ & (0,1), && (0,2), \\
& (1,1), && (1,2), \\
& (2,1), && (2,2), \\
& (3,1), && (3,2), \\
& (4,1), && (4,2), \\
& && \vdots \\
& (k,1), && (k,2), \\
& && \vdots \qquad \}
\end{aligned}
$$

Cartesian product

<u>Ex</u> $\mathbb{N} \times \mathbb{N} = \{ (a,b) \mid a \in \mathbb{N},\ b \in \mathbb{N} \}$

$$
\begin{aligned}
= \{ &(0,0), && (0,1), && (0,2), && (0,3), && \ldots \\
&(1,0), && (1,1), && (1,2), && (1,3), && \ldots \\
&(2,0), && (2,1), && (2,2), && && \\
&(3,0), && (3,1), && (3,2), && && \\
&(4,0), && (4,1), && (4,2), && && \\
& && \vdots && && && \ddots \quad \}
\end{aligned}
$$

Visualize as "integer points" on the 1st quadrant :



Cartesian product

Def The <u>Cartesian product</u> of two sets A and B is $A \times B = \{ (a,b) \mid a \in A, b \in B \}$

Ex Let $A = \{1, 2, 3\}$.

What is $A \times A$?

$\{ (1,1), (1,2), \ldots$

Note: $|A \times A| = ?$ $|A| \cdot |A| = 3 \cdot 3 = 9$

Cartesian product

<u>Def</u> The <u>Cartesian product</u> of two sets A and B

is $$A \times B = \{ (a,b) \mid a \in A, \ b \in B \}$$

<u>Fact</u> In general, if A and B are finite sets, then

$$|A \times B| = |A| \times |B|$$

<u>Ex</u> $$|\{1,2,3\} \times \{\heartsuit, \square\}| = 3 \cdot 2 = 6$$

$$\{1,2,3\} \times \{\heartsuit\} = \{(1,\heartsuit), (2,\heartsuit), (3,\heartsuit)\}$$

$$A \times \varnothing = \varnothing$$

Cartesian product

<u>Def</u> The <u>Cartesian product</u> of two sets A and B

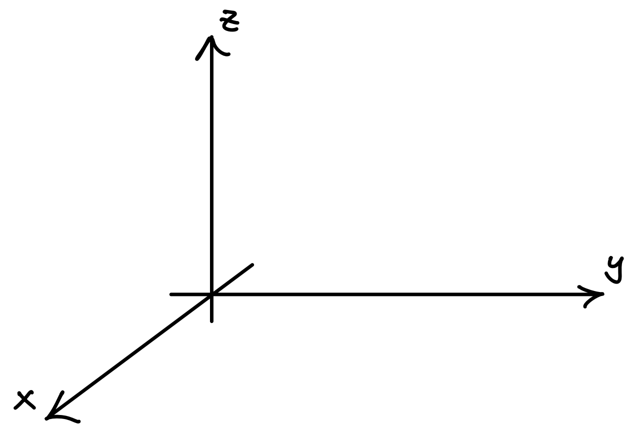
is $A \times B = \{ (a,b) \mid a \in A, \ b \in B \}$

<u>Def</u> The <u>Cartesian product</u> of three sets $A, B,$ and C

is $A \times B \times C = \{ (a,b,c) \mid a \in A, \ b \in B, c \in C \}$

<u>Ex</u> $\underbrace{\mathbb{R} \times \mathbb{R} \times \mathbb{R}}_{\text{or } \mathbb{R}^3} = \{ (x,y,z) \mid x, y, z \in \mathbb{R} \}$

The elements can be visualized as all points in the xyz-space



Cartesian product

<u>Def</u> The <u>Cartesian product</u> of k sets $A_1, A_2, \dots, A_k$ is $A_1 \times A_2 \times \dots \times A_k = \{(a_1, a_2, \dots, a_k) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_k \in A_k\}$

<u>Ex</u> $\{\star\} \times \{1, 2\} \times \{\heartsuit, \square\} \times \{2\} = $?

Cartesian product

<u>Def</u> The <u>Cartesian product</u> of k sets $A_1, A_2, \ldots, A_k$

is $A_1 \times A_2 \times \ldots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_1 \in A_1,\ a_2 \in A_2, \ldots, a_k \in A_k\}$

<u>Ex</u> $\{\star\} \times \{1,2\} \times \{\heartsuit, \square\} \times \{2\} =$

$\{ (\star, 1, \heartsuit, 2), (\star, 1, \square, 2),$
$(\star, 2, \heartsuit, 2), (\star, 2, \square, 2) \}$

<u>Note</u> The cardinality here is $1 \cdot 2 \cdot 2 \cdot 1 = 4$

In general, $|A_1 \times \ldots \times A_k| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_k|$

Cartesian product

If the sets in a Cartesian product are the same, it's common to use exponents:

$$A^2 = A \times A$$
$$A^3 = A \times A \times A$$

In general, $A^k = \underbrace{A \times A \times \dots \times A}_{k \text{ factors}}$

Ex $\{0,1\}^3$ is

Cartesian product

If the sets in a Cartesian product are the same, it's common to use exponents:

$$A^2 = A \times A$$
$$A^3 = A \times A \times A$$

In general, $A^k = \underbrace{A \times A \times \dots \times A}_{k \text{ factors}}$

<u>Ex</u> $\{0,1\}^3$ is $\{0,1\} \times \{0,1\} \times \{0,1\} =$

These four correspond to $\{0,1\} \times \{0,1\}$

$\{(0,0,0), (0,0,1), (0,1,0), (0,1,1),$
$(1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

<u>Note</u>: I sorted the elements using <u>lexicographic</u> order (like in a dictionary)

Pseudocode

Pseudocode is a hybrid between explaining something to a person and explaining it to a computer.

To read pseudocode: pretend you are a computer.

To write pseudocode: imagine explaining something
to a very literal-minded person
with no short-term memory
who has to write everything down.

Ex pseudocode with a loop:

```
for a from 1 to 3 do:
    print a, a+1
```

output:
```
1, 2
2, 3
3, 4
```

Pseudocode

__Ex__ pseudocode with nested loop:

```
for a from 0 to 1 do:
    for b from 0 to 1 do:
        print a, b
```

output:

Pseudocode

Ex pseudocode with nested loop:

```
for a from 0 to 1 do:
    for b from 0 to 1 do:
        print a, b
```

"outer loop" executes the inner loop twice

Within this "inner loop", a is fixed and b ranges from 0 to 1

output: 0, 0
0, 1

first time through the outer loop, a = 0

1, 0
1, 1

Second time through the outer loop, a = 1

Pseudocode

Ex A pseudocode:

```
for a from 0 to 1 do:
    for b from 0 to 1 do:
        for c from 0 to 1 do:
            print (a,b,c)
```

The output is the ordered triples

$$(0,0,0),\ (0,0,1),\ (0,1,0),\ (0,1,1),$$
$$(1,0,0),\ (1,0,1),\ (1,1,0),\ (1,1,1),$$

i.e. the elements of the Cartesian product $\{0,1\}^3 = \{0,1\} \times \{0,1\} \times \{0,1\}$ written in lexicographic (aka dictionary) order.

Power set

<u>Recall</u> The set S is a <u>subset</u> of T (notation: $S \subseteq T$) if every element of S is an element of T

<u>Ex</u> What are the subsets of $\{a, b, c\}$?

$\{\overset{0}{a}, \overset{0}{b}, \overset{0}{c}\}$ $\quad$ $\{\overset{0}{a}, \overset{0}{b}, \overset{1}{c}\}$ $\quad$ $\{\overset{0}{a}, \overset{1}{b}, \overset{0}{c}\}$ $\quad$ $\{\overset{0}{a}, \overset{1}{b}, \overset{1}{c}\}$

$\{\overset{1}{a}, \overset{0}{b}, \overset{0}{c}\}$ $\quad$ $\{\overset{1}{a}, \overset{0}{b}, \overset{1}{c}\}$ $\quad$ $\{\overset{1}{a}, \overset{1}{b}, \overset{0}{c}\}$ $\quad$ $\{\overset{1}{a}, \overset{1}{b}, \overset{1}{c}\}$

Method 1: List in a "lexicographic" order

Power set

<u>Recall</u> The set S is a <u>subset</u> of T (notation: $S \subseteq T$) if every element of S is an element of T

<u>Ex</u> What are the subsets of $\{a, b, c\}$?

$$\{\overset{0}{\cancel{a}}, \overset{0}{\cancel{b}}, \overset{0}{\cancel{c}}\} \qquad \{\overset{0}{\cancel{a}}, \overset{0}{\cancel{b}}, \overset{1}{c}\} \qquad \{\overset{0}{\cancel{a}}, \overset{1}{b}, \overset{0}{\cancel{c}}\} \qquad \{\overset{0}{\cancel{a}}, \overset{1}{b}, \overset{1}{c}\}$$

$$\varnothing \qquad \{c\} \qquad \{b\} \qquad \{b, c\}$$

$$\{\overset{1}{a}, \overset{0}{\cancel{b}}, \overset{0}{\cancel{c}}\} \qquad \{\overset{1}{a}, \overset{0}{\cancel{b}}, \overset{1}{c}\} \qquad \{\overset{1}{a}, \overset{1}{b}, \overset{0}{\cancel{c}}\} \qquad \{\overset{1}{a}, \overset{1}{b}, \overset{1}{c}\}$$

$$\{a\} \qquad \{a, c\} \qquad \{a, b\} \qquad \{a, b, c\}$$

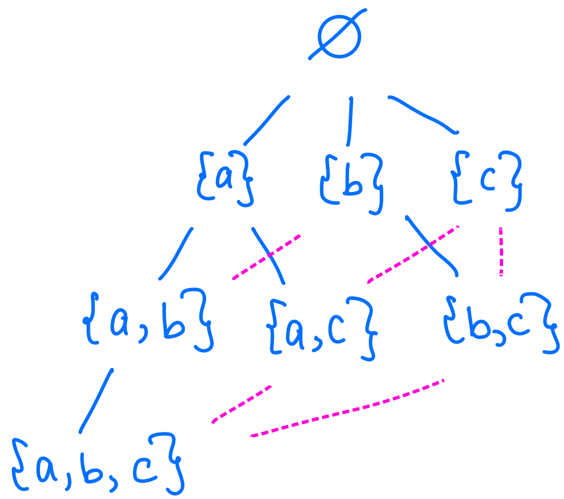
Answer: $\varnothing$, $\{c\}$, $\{b\}$, $\{b, c\}$,
$\{a\}$, $\{a, c\}$, $\{a, b\}$, $\{a, b, c\}$

Power set

Recall The set S is a subset of T (notation: $S \subseteq T$)
if every element of S is an element of T

Ex What are the subsets of $\{a, b, c\}$?

Alternatively, draw a tree (Method 2):

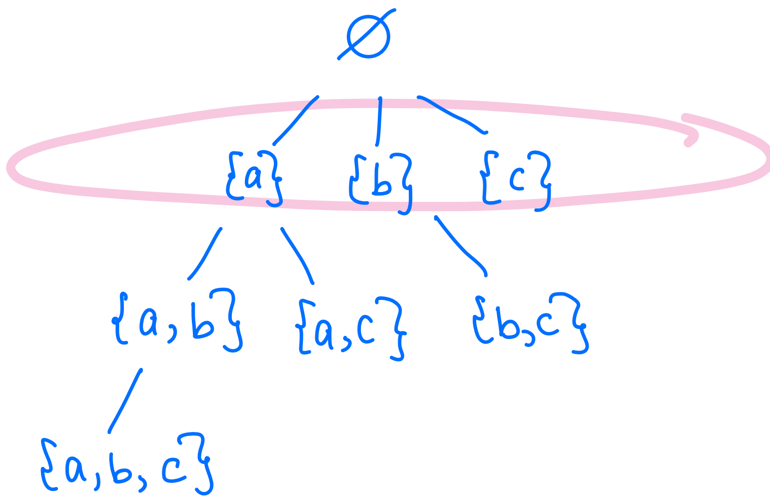


You need to be able to use both methods

Power set

<u>Recall</u> The set S is a <u>subset</u> of T (notation: $S \subseteq T$) if every element of S is an element of T

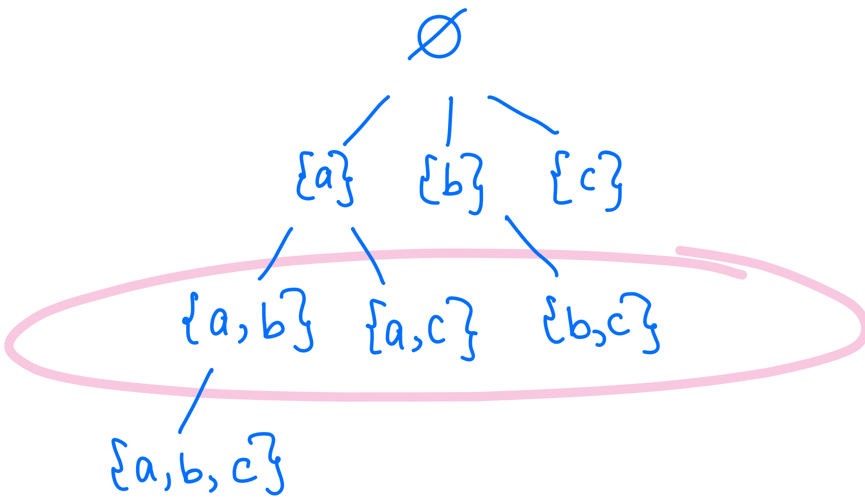
<u>Ex</u> Find all $\underbrace{\text{1-element subsets}}_{\text{called "singletons"}}$ of $\{a, b, c\}$



Power set

__Recall__ The set S is a __subset__ of T (notation: $S \subseteq T$) if every element of S is an element of T

__Ex__ Find all 2-element subsets of $\{a, b, c\}$



Power set (Extra)

A pseudocode that lists all two-element subsets in lexicographic order, listing each subset only once.

$\underbrace{\text{pairs}}_{}$ name of my pseudocode

```
pairs(S) :=
    for a in S do:
        for all b after a in S do:
            output({a,b})
```

Ex If my input S is $S = \{1, 2, 3\}$,

the program would output $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, in this order

Power set

Recall The set S is a subset of T (notation: $S \subseteq T$) if every element of S is an element of T

Def The power set of a set T is

$$\mathcal{P}(T) = \{\text{all subsets of } T\}$$

Note The two "extreme" subsets, $\emptyset$ and T, are elements of $\mathcal{P}(T)$.

Ex The power set of $\{a,b,c\}$ is

$$\mathcal{P}(\{a,b,c\}) = \{\emptyset, \{c\}, \{b\}, \{b,c\}, \{a\}, \{a,c\}, \{a,b\}, \{a,b,c\}\}$$

Note 1. Is $\{a\}$ an element of $\mathcal{P}(\{a,b,c\})$? Yes

2. Is a an element of $\mathcal{P}(\{a,b,c\})$? NO

Questions

Let $x = \{1\}$ and $T = \{1, \{1\}\}$

a. Is $x \in T$? Yes

b. Is $x \subseteq T$? Yes

Exercise #1

Let $A = \{0, 2, 3\}$, $B = \{2, 3\}$, and $C = \{1, 4\}$ be subsets of the universe $U = \{0, \ldots, 4\}$.

Compute the following.

1. $A \times A^c$

2. $B \times \mathcal{P}(B)$

3. $U \times \varnothing$

Exercise #1

Let $A = \{0, 2, 3\}$, $B = \{2, 3\}$, and $C = \{1, 4\}$ be subsets of the universe $U = \{0, \ldots, 4\}$.

Compute the following.

1. $A \times A^c = \{0, 2, 3\} \times \{1, 4\} = \{(0,1), (0,4), (2,1), (2,4), (3,1), (3,4)\}$

2. $B \times \mathcal{P}(B) = \{2, 3\} \times \{\emptyset, \{2\}, \{3\}, \{2,3\}\} = \{(2,\emptyset), (2,\{2\}), (2,\{3\}), (2,\{2,3\}), (3,\emptyset), (3,\{2\}), (3,\{3\}), (3,\{2,3\})\}$

3. $U \times \emptyset = \emptyset$

Exercise #2

Suppose you are playing a game.
When it's your turn, you flip a quarter then roll a die.

1. We can model this situation using Cartesian product

$$\{\quad ? \quad\} \times \{\quad ? \quad\}$$

2. The set of all possible outcomes is ?

3. The number of possible outcomes is ?

Exercise

Suppose you are playing a game.
When it's your turn, you flip a quarter then roll a die.

1. We can model this situation using Cartesian product

$$\{\text{HEADS}, \text{TAILS}\} \times \{1, 2, 3, 4, 5, 6\}$$

Set $A :=$ and $B :=$

2. The set of all possible outcomes is $A \times B$

3. The number of possible outcomes is $|A \times B| = |A| \cdot |B|$
$= 2 \cdot 6$
$= 12$

— the end —

Next Topic:

Sec 1.4 Binary Representation