Sec 8.4 Hyperplanes

Def (from Sec 8.1) A line in $\mathbb{R}^n$ is a flat of dimension 1

A hyperplane in $\mathbb{R}^n$ is a flat of dimension $n-1$.

Ex A hyperplane in $\mathbb{R}^2$ is a line.

An implicit equation of a line in $\mathbb{R}^2$ has the form $\underbrace{ax+by}_{\text{a linear expression}} = d$

Ex A hyperplane in $\mathbb{R}^3$ is a plane.

An implicit equation of a plane in $\mathbb{R}^3$ has the form $\underbrace{ax+by+cz}_{\text{a linear expression}} = d$.

So a plane is the set of all points at which a linear expression has a fixed value, d.

Def
- A linear functional on $\mathbb{R}^n$ is a linear map $f : \mathbb{R}^n \to \mathbb{R}$.
- If d is a scalar in $\mathbb{R}$, then the symbol

$$[f : d]$$

denotes the set of all $\vec{x}$ in $\mathbb{R}^n$ at which the value of f is d.

That is,

$$[f : d] = \{ \vec{x} \in \mathbb{R}^n : f(\vec{x}) = d \}.$$

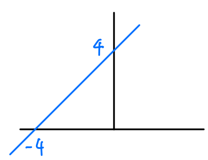
- The zero functional is the map $f(\vec{x}) = 0$ for all $\vec{x}$ in $\mathbb{R}^n$.

All other linear functionals on $\mathbb{R}^n$ are called nonzero.

Ex The line $x - y = 4$ in $\mathbb{R}^2$ is a hyperplane in $\mathbb{R}^2$.

It is equal to $[f : 4]$ where f is the linear functional $f(x,y) = x - y$,

i.e. the set of points in $\mathbb{R}^2$ at which $f(x,y)$ has the value 4.



If f is a linear functional on $\mathbb{R}^n$, then the standard matrix of this linear transformation f is a $1 \times n$ matrix A, say $A = [\, a_1 \quad a_2 \quad \cdots \quad a_n \,]$. So

$$[f : 0] \quad \text{is the same as} \quad \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = 0\} = \text{Nul}\, A \tag{1}$$

If f is a nonzero functional, then rank $A = 1$, and dim Nul $A = n - 1$, by the Rank Theorem.[2] Thus, the subspace $[f : 0]$ has dimension $n - 1$ and so is a hyperplane. Also, if d is any number in $\mathbb{R}$, then

$$[f : d] \quad \text{is the same as} \quad \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = d\} \tag{2}$$

Recall from Theorem 6 in Section 1.5 that the set of solutions of $A\mathbf{x} = \mathbf{b}$ is obtained by translating the solution set of $A\mathbf{x} = \mathbf{0}$, using any particular solution $\mathbf{p}$ of $A\mathbf{x} = \mathbf{b}$. When A is the standard matrix of the transformation f, this theorem says that

$$[f : d] = [f : 0] + \mathbf{p} \quad \text{for any } \mathbf{p} \text{ in } [f : d] \tag{3}$$

Thus the sets $[f : d]$ are hyperplanes parallel to $[f : 0]$. See Figure 1.

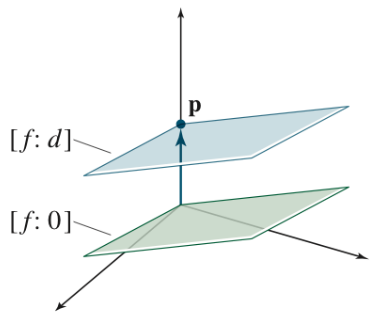


FIGURE 1 Parallel hyperplanes, with $f(\mathbf{p}) = d$.

When A is a $1 \times n$ matrix, the equation $A\mathbf{x} = d$ may be written with an inner product $\mathbf{n} \cdot \mathbf{x}$, using $\mathbf{n}$ in $\mathbb{R}^n$ with the same entries as A. Thus, from (2),

$$[f : d] \quad \text{is the same as} \quad \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot \mathbf{x} = d\} \tag{4}$$

[2] See Theorem 14 in Section 2.9 or Theorem 14 in Section 4.5.

Then $[f : 0] = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot \mathbf{x} = 0\}$, which shows that $[f : 0]$ is the orthogonal complement of the subspace spanned by $\mathbf{n}$. In the terminology of calculus and geometry for $\mathbb{R}^3$, $\mathbf{n}$ is called a **normal** vector to $[f : 0]$. (A "normal" vector in this sense need not have unit length.) Also, $\mathbf{n}$ is said to be **normal** to each parallel hyperplane $[f : d]$, even though $\mathbf{n} \cdot \mathbf{x}$ is not zero when $d \neq 0$.

EX 3: Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; \vec{n} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 12 \right\}$ where $\vec{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a) Find an implicit equation of H (i.e. a linear equation $ax+by=c$)

Sol: $\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 12$, so an implicit equation of H is $\boxed{3x + 4y = 12}$

b) Find a linear functional f and a scalar d such that $H = [f : d]$

Sol: $f(x,y) = 3x + 4y$ and $d = 12$

c) Let $\vec{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$. Consider the parallel hyperplane (line) $H_1 = H + \vec{v}$.

Find an implicit description.

Sol: H_1 will be of the form $[f : d_1]$ for some d_1 in $\mathbb{R}$.

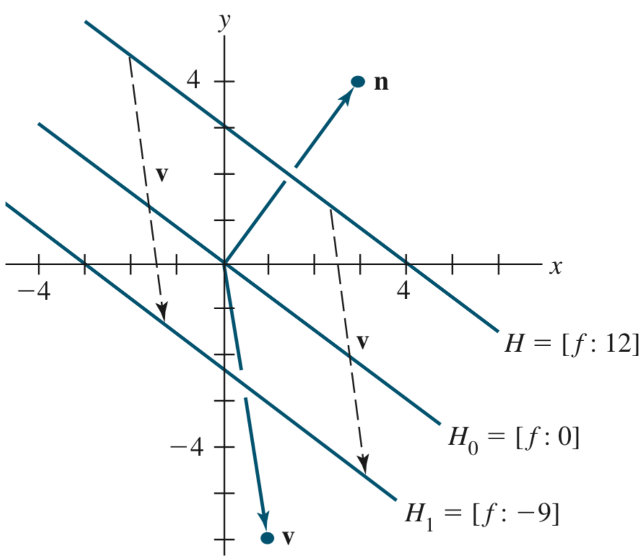
To find d_1, first find a point $\vec{p}$ in H_1

(by taking a point in H and adding $\vec{v}$ to it).

An easy point in H to work with is $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

Let $\vec{p} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

Compute $\vec{n} \cdot \vec{p} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = 3 - 12 = -9$. Then $d_1 = -9$, so $H_1 = [f : -9]$.



$3x + 4y = 12$

$3x + 4y = 0$

$3x + 4y = -9$

d) Give an explicit description of the line $3x+4y=12$ in parametric vector form, i.e. of form $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{v}_1 + t\,\vec{v}_2$, t in $\mathbb{R}$.

Sol: Solve the equation $\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \end{bmatrix}$

$A \quad \bar{x} = \bar{b}$

The matrix $\begin{bmatrix} 3 & 4 \end{bmatrix}$ is already in (row) echelon form.

Let y be the free variable.

$3x+4y=12 \Rightarrow \quad x = \frac{12}{3} - \frac{4}{3}y$

The solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 + \frac{4}{3}y \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + y \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$, y in $\mathbb{R}$

An explicit description of $3x+4y=12$ in parametric form is

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}, \text{ for } t \text{ in } \mathbb{R}}$$

Converting an explicit description of a line into implicit form is more involved. The basic idea is to construct $[f:0]$ and then find d for $[f:d]$.

(Extra ex)

EXAMPLE 5 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$, and let L_1 be the line through $\mathbf{v}_1$ and $\mathbf{v}_2$. Find a linear functional f and a constant d such that $L_1 = [f:d]$.

SOLUTION The line L_1 is parallel to the translated line L_0 through $\mathbf{v}_2 - \mathbf{v}_1$ and the origin. The defining equation for L_0 has the form

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \text{or} \quad \mathbf{n}\cdot\mathbf{x} = 0, \quad \text{where} \quad \mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{5}$$

Since $\mathbf{n}$ is orthogonal to the subspace L_0, which contains $\mathbf{v}_2 - \mathbf{v}_1$, compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 6 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

and solve

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = 0$$

By inspection, a solution is $\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 2 & 5 \end{bmatrix}$. Let $f(x,y) = 2x + 5y$. From (5), $L_0 = [f:0]$, and $L_1 = [f:d]$ for some d. Since $\mathbf{v}_1$ is on line L_1, $d = f(\mathbf{v}_1) = 2(1) + 5(2) = 12$. Thus, the equation for L_1 is $2x + 5y = 12$. As a check, note that $f(\mathbf{v}_2) = f(6,0) = 2(6) + 5(0) = 12$, so $\mathbf{v}_2$ is on L_1, too. ■

Ex 6 Let H_1 be the plane that contains the points $\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}2\\-1\\4\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}3\\1\\2\end{bmatrix}$.
Find an implicit description $[f : d]$ of H_1.

Note: Since this is the special case of $\mathbb{R}^3$, you can use the cross-product formula (from Calc 3) to compute a normal vector for the plane.
But we'll use a procedure below that generalizes to higher dimension.

Sol:

(Step 1) Recall (from Sec 8.1 "Affine combinations" Thm 1)

$$H_1 \text{ is parallel to } H_0 = \operatorname{Span}\Big\{ \underbrace{\vec{v}_2 - \vec{v}_1}_{\text{denote } \vec{p}},\ \underbrace{\vec{v}_3 - \vec{v}_1}_{\text{denote } \vec{q}} \Big\} = \operatorname{Span}\Big\{ \underbrace{\begin{bmatrix}1\\-2\\3\end{bmatrix}}_{\vec{p}},\ \underbrace{\begin{bmatrix}2\\0\\1\end{bmatrix}}_{\vec{q}} \Big\}$$

Note H_0 is a subspace, so H_0 is the plane containing $\vec{p}$, $\vec{q}$, and the origin.

(Step 2) Find a vector $\vec{n} = \begin{bmatrix}a\\b\\c\end{bmatrix}$ which is normal to H_0:

Recall (from Sec 6.1):
A vector $\vec{z}$ in $\mathbb{R}^n$ is said to be <u>orthogonal</u> to a subspace W of $\mathbb{R}^n$ if $\vec{z}$ is orthogonal to every vector in W, but it's sufficient to check that $\vec{z}$ is orthogonal to every vector in a spanning set of W.

Let's find a nonzero $\vec{n}$ which is orthogonal to both $\vec{p}$ and $\vec{q}$.

$$\left.\begin{array}{lll} \text{Set} & \vec{p}\cdot\vec{n} = 0 : & 1a - 2b + 3c = 0 \\ \text{Set} & \vec{q}\cdot\vec{n} = 0 : & 2a \qquad\ + c = 0 \end{array}\right\} \text{ Solve for } a, b, c$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \overset{\text{Row reduce}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 4 & -5 & 0 \end{array}\right] \quad \left.\begin{array}{l} 2a \quad + c = 0 \\ 4b - 5c = 0 \end{array}\right\} \quad \begin{array}{l} 2a = -c \\ 4b = 5c \end{array} \quad c \text{ is free}$$

Choose any nonzero c, let's say $c = 4$. So $a = -\frac{4}{2} = -2$, $b = \frac{5(4)}{4} = 5$

Let $\vec{n} = \begin{bmatrix}-2\\5\\4\end{bmatrix}$. Then $H_0 = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \middle| \begin{pmatrix}-2\\5\\4\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0 \right\}$, i.e. $H_0 = [f : 0]$
where $f(x,y,z) = -2x + 5y + 4z$.

(Step 3) Find d where $H_1 = [f : d]$.

Simply choose a point in H_1 and input it into f:
We know $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are in H_1. Set $d = f(\vec{v}_1) = -2(1) + 5(1) + 4(1) = 7$.
Check: $f(\vec{v}_2) = f(\vec{v}_3) = 7$.

Then H_1 is $[f : 7]$ where $f(x,y,z) = -2x + 5y + 4z$. or $\boxed{-2x + 5y + 4z = 7}$

Thm (Thm 11) Suppose H is a subset of $\mathbb{R}^n$.
H is a hyperplane iff
$H = [f : d]$ for some nonzero linear functional f and some d in $\mathbb{R}$ iff
$H = \{\vec{x} \text{ in } \mathbb{R}^n \mid \vec{n} \cdot \vec{x} = d\}$ for some nonzero vector $\vec{n}$ and some d in $\mathbb{R}$

Many important applications of hyperplanes depend on the possibility of "separating" two sets by a hyperplane. Intuitively, this means that one of the sets is on one side of the hyperplane and the other set is on the other side. The following terminology and notation will help to make this idea more precise.

TOPOLOGY IN $\mathbb{R}^n$: TERMS AND FACTS

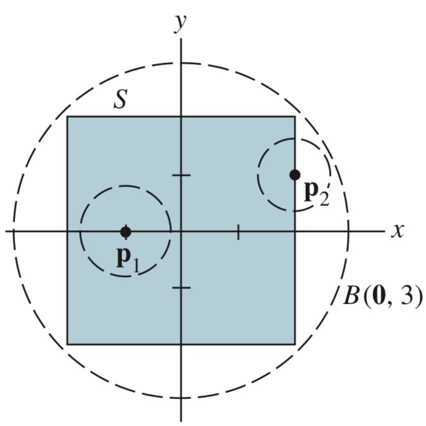
For any point $\mathbf{p}$ in $\mathbb{R}^n$ and any real $\delta > 0$, the **open ball** $B(\mathbf{p}, \delta)$ with center $\mathbf{p}$ and radius δ is given by

$$B(\mathbf{p}, \delta) = \{\mathbf{x} : \|\mathbf{x} - \mathbf{p}\| < \delta\}$$

Given a set S in $\mathbb{R}^n$, a point $\mathbf{p}$ is an **interior point** of S if there exists a $\delta > 0$ such that $B(\mathbf{p}, \delta) \subseteq S$. If every open ball centered at $\mathbf{p}$ intersects both S and the complement of S, then $\mathbf{p}$ is called a **boundary point** of S. A set is **open** if it contains none of its boundary points. (This is equivalent to saying that all of its points are interior points.) A set is **closed** if it contains all of its boundary points. (If S contains some but not all of its boundary points, then S is neither open nor closed.) A set S is **bounded** if there exists a $\delta > 0$ such that $S \subseteq B(\mathbf{0}, \delta)$. A set in $\mathbb{R}^n$ is **compact** if it is closed and bounded.

EXAMPLE 7 Let

$$S = \text{conv}\left\{\begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}, \quad \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{p}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$



$\vec{p}_1$ is an interior point of S

$\vec{p}_2$ is a boundary point of S

S is bounded by $B(\vec{0}, 3)$

S is closed

So S is compact

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Notation: If f is a linear functional, then $f(A) \le d$ means $f(\mathbf{x}) \le d$ for each $\mathbf{x} \in A$.

$f(A) < d$ means $f(\vec{x}) < d$ for each $\vec{x}$ in A

DEFINITION

The hyperplane $H = [f:d]$ **separates** two sets A and B if one of the following holds:

(i) $f(A) \le d$ and $f(B) \ge d$, or

(ii) $f(A) \ge d$ and $f(B) \le d$.

If in the conditions above all the weak inequalities are replaced by strict inequalities, then H is said to **strictly separate** A and B.

8.5 Polytopes

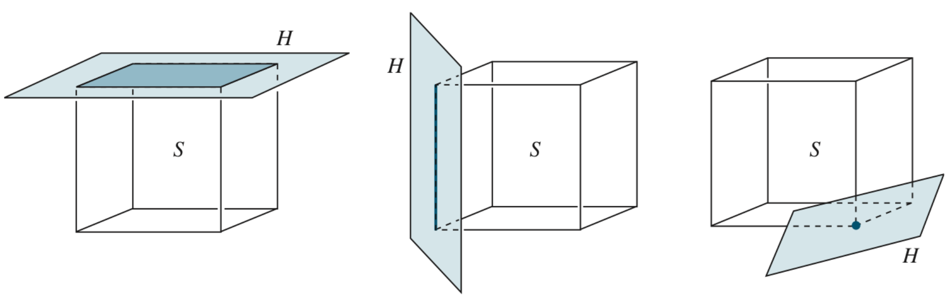
A **polytope** in $\mathbb{R}^n$ is the convex hull of a finite set of points. In $\mathbb{R}^2$, a polytope is simply a polygon. In $\mathbb{R}^3$, a polytope is called a polyhedron. Important features of a polyhedron are its faces, edges, and vertices. For example, the cube has 6 square faces, 12 edges, and 8 vertices.

EXAMPLE 1 Suppose S is a cube in $\mathbb{R}^3$. When a plane H is translated through $\mathbb{R}^3$ until it just touches (supports) the cube but does not cut through the interior of the cube, there are three possibilities for $H \cap S$, depending on the orientation of H. (See Figure 1.)

$H \cap S$ may be a 2-dimensional square face (facet) of the cube.

$H \cap S$ may be a 1-dimensional edge of the cube.

$H \cap S$ may be a 0-dimensional vertex of the cube. ■



$H \cap S$ is 2-dimensional. $H \cap S$ is 1-dimensional. $H \cap S$ is 0-dimensional.