Sec 8.3 Convex combinations

(Recall from Sec 8.1)

Def • An <u>affine combination</u> of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ in $\mathbb{R}^n$

is a linear combination $c_1\vec{v}_1 + \dots + c_p\vec{v}_p$

such that $\underbrace{c_1 + c_2 + \dots + c_p = 1}_{\text{the weights sum up to 1}}$

• The <u>affine hull</u> (or <u>affine span</u>) of a set S

is the set of all affine combinations of points in S.

(Notation: aff S)

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such that $\underbrace{c_1 + c_2 + \dots + c_p = 1}_{\text{the weights sum up to 1}}$

<u>AND</u> all $c_1, c_2, \dots, c_p$ are nonnegative

• The <u>convex</u> <u>hull</u> of a set S is the set of all convex combinations of points in S. (Notation: Conv S)

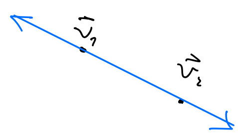
Note: • aff $\{\vec{v}\}$ is just $\{\vec{v}\}$,

conv $\{\vec{v}\}$ is also $\{\vec{v}\}$

• aff $\{\vec{v}_1, \vec{v}_2\}$ is $\{ \overbrace{(1-t)\vec{v}_1 + t\vec{v}_2}^{\text{affine combinations of } \vec{v}_1 \text{ and } \vec{v}_2} \mid t \text{ in } \mathbb{R}\}$,

(distinct points)

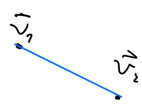
the line containing both $\vec{v}_1, \vec{v}_2$.



conv $\{\vec{v}_1, \vec{v}_2\}$ is $\{ \overbrace{(1-t)\vec{v}_1 + t\vec{v}_2}^{\text{convex combinations of } \vec{v}_1 \text{ and } \vec{v}_2} \mid t \text{ between 0 and 1}\}$,

Because the weights in a convex combination are nonnegative

the line segment between $\vec{v}_1$ and $\vec{v}_2$, denoted $\overline{\vec{v}_1\vec{v}_2}$.



Recall (Sec 8.2 "Unique representation thm" & "barycentric coordinates")

If S is affinely independent and $\vec{p}$ is in aff S

then $\vec{p}$ can be written uniquely as an affine combination of S.

The weights are called barycentric coordinates of $\vec{p}$.

Note: $\vec{p}$ is in conv S iff the barycentric coordinates of $\vec{p}$ are nonnegative.

EXAMPLE 1 Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{p}_1 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} -10 \\ 5 \\ 11 \\ -4 \end{bmatrix},$$

Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ & note S is an orthogonal set.

Determine whether $\vec{p}_1, \vec{p}_2$ are in Span S, aff S, & conv S.

Sol: Let W = Span S

Recall that $\text{proj}_W \vec{p}$ is the closest point in W to $\vec{p}$,

so $\vec{p}$ is in W iff $\text{proj}_W \vec{p} = \vec{p}$.

For $\vec{p}_1$:

$$\text{proj}_W \vec{p}_1 = \text{proj}_{\vec{v}_1} \vec{p}_1 + \text{proj}_{\vec{v}_2} \vec{p}_1 + \text{proj}_{\vec{v}_3} \vec{p}_1$$

(We can do this because $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis of W, see Sec 6.2 & 6.3)

$$= \frac{\mathbf{p}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{p}_1 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{p}_1 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3$$

$$= \frac{18}{54}\mathbf{v}_1 + \frac{18}{54}\mathbf{v}_2 + \frac{18}{54}\mathbf{v}_3$$

$$= \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 0 \end{bmatrix} = \mathbf{p}_1$$

This shows $\vec{p}_1$ is in W = Span S

Since the coefficients sum up 1, $\vec{p}_1$ is in aff S.

——— " ——— and are nonnegative, $\vec{p}_1$ is in conv S.

For $\vec{p}_2$:
$$\text{proj}_W \vec{p}_2 = \frac{\vec{p}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{p}_2 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \frac{\vec{p}_2 \cdot \vec{v}_3}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3$$
$$= \frac{48}{54}\vec{v}_1 + \frac{108}{54}\vec{v}_2 + \frac{-12}{54}\vec{v}_3$$
$$= \frac{8}{9}\begin{bmatrix} 3 \\ 0 \\ 6 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -6 \\ 3 \\ 3 \\ 0 \end{bmatrix} - \frac{2}{9}\begin{bmatrix} 3 \\ 6 \\ 0 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} * \\ 0 + 2(3) - \frac{12}{9} \\ * \\ * \end{bmatrix} \neq \vec{p}_2$$

So $\vec{p}_2$ is not in $W = \text{Span } S$.

Thus $\vec{p}_2$ cannot be in aff S or conv S.

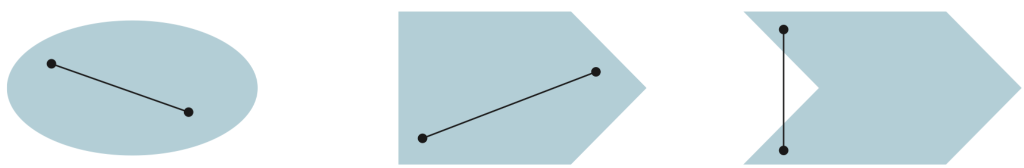
Recall (Sec 8.1): A set S is affine if:

$\vec{p}, \vec{q}$ in S implies the entire line through $\vec{p}$ & $\vec{q}$ is in S.

DEFINITION

A set S is **convex** if for each $\mathbf{p}, \mathbf{q} \in S$, the line segment $\overline{\mathbf{pq}}$ is contained in S.

Intuitively, a set S is convex if every two points in the set can "see" each other without the line of sight leaving the set. Figure 1 illustrates this idea.



Convex Convex Not convex

THEOREM 7

A set S is convex if and only if every convex combination of points of S lies in S. That is, S is convex if and only if $S = \text{conv } S$.

THEOREM 8

The intersection of two subspaces is a subspace
— " — of two convex sets is a convex set
— " — of two affine sets is an affine set

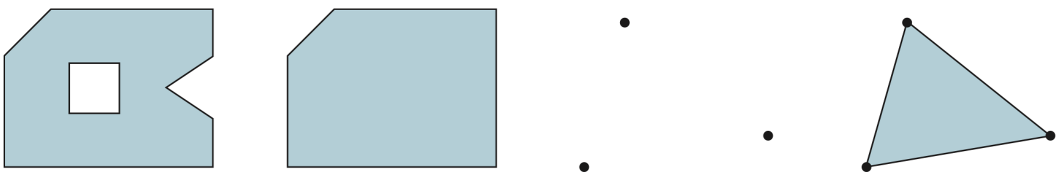
THEOREM 9

For any set S, the convex hull of S is the intersection of all the convex sets that contain S.

i.e. conv S is the "smallest" convex set containing S.

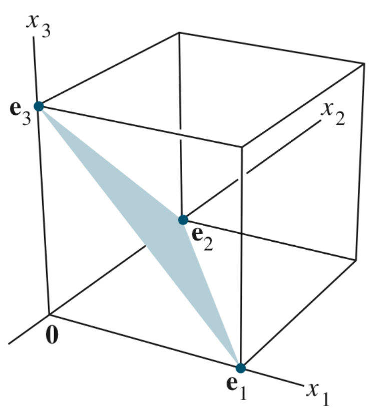
EXAMPLE 2

a. The convex hulls of sets S and T in $\mathbb{R}^2$ are shown below.



S conv S T conv T

b. Let S be the set consisting of the standard basis for $\mathbb{R}^3$, $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then conv S is a triangular surface in $\mathbb{R}^3$, with vertices $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. See Figure 2. ■



If $\vec{p}$ is in the convex hull of S, then (by def) $\vec{p}$ is a convex combination of points of S.

How many points are needed for this combination?

THEOREM 10

(Caratheodory) If S is a nonempty subset of $\mathbb{R}^n$, then every point in conv S can be expressed as a convex combination of $n+1$ or fewer points of S.

Illustration of proof (using an example)

EXAMPLE 4 Let

(Here $n=2$)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and } S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}.$$

Let $$\mathbf{p} = \tfrac{1}{4}\mathbf{v}_1 + \tfrac{1}{6}\mathbf{v}_2 + \tfrac{1}{2}\mathbf{v}_3 + \tfrac{1}{12}\mathbf{v}_4 = \begin{bmatrix} \frac{10}{3} \\ \frac{5}{2} \end{bmatrix}$$

Use the procedure in the proof of Caratheodory's Theorem to express **p** as a convex combination of $\underbrace{\text{three}}_{n+1}$ points of S.

The set $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4\}$ of homogeneous forms in $\mathbb{R}^3$ is lin. dependent (since every basis of $\mathbb{R}^3$ has 3 elts), so by Thm 5 (Sec 8.2) S is affinely dependent.

Find an affine dependence relation $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (using Sec 8.2):

First, find a linear dependence relation $c_1\tilde{v}_1 + c_2\tilde{v}_2 + c_3\tilde{v}_3 + c_4\tilde{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$:

$$M = \begin{array}{c} \begin{array}{cccc} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{v}_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 5 & 3 \\ 0 & 3 & 4 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \xrightarrow{\text{row reduce}} \begin{array}{c} \begin{array}{cccc} \underbrace{}_{\text{pivot cols}} & \text{free column} \end{array} \\ \begin{bmatrix} 4 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 4 & 3 \end{bmatrix} \end{array}$$

$4c_1 + 5c_4 = 0$

$c_2 - c_4 = 0$

$4c_3 + 3c_4 = 0$

c_4 can be any number.

Choose any nonzero number for c_4, say $c_4 = 4$

Then $4c_1 = -5(4) \Rightarrow c_1 = -5$

$c_2 = 4$

$4c_3 = -3(4) \Rightarrow c_3 = -3$

So $-5v_1 + \underset{\text{pos}}{4}v_2 - 3v_3 + \underset{\text{pos}}{4}v_4 = \begin{pmatrix}0\\0\end{pmatrix}$ is an affine dependence of S

Affine dependence means the coefficients sum to 0, so some will be positive.

Take the points v_2 and v_4 (whose coeffs are positive).

Compute the ratio of the coefficients in eq for $\vec{p}$:

$$\frac{1}{4}\mathbf{v}_1 + \frac{1}{6}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 + \frac{1}{12}\mathbf{v}_4 = \begin{bmatrix}\frac{10}{3}\\ \frac{5}{2}\end{bmatrix}$$

and $-5v_1 + 4v_2 - 3v_3 + 4v_4 = \begin{pmatrix}0\\0\end{pmatrix}$ the affine dependence relation.

For v_2: $\dfrac{\left(\frac{1}{6}\right)}{4} = \frac{1}{6}\cdot\frac{1}{4} = \frac{1}{24}$ For v_4: $\dfrac{\left(\frac{1}{12}\right)}{4} = \frac{1}{12}\cdot\frac{1}{4} = \frac{1}{48}$

$\frac{1}{48}$ is the smaller of these

Subtract:

$$\frac{1}{4}v_1 + \frac{1}{6}v_2 + \frac{1}{2}v_3 + \frac{1}{12}v_4 = \begin{pmatrix}10/3\\5/2\end{pmatrix}$$

$$\frac{1}{48}\left(-5v_1 + 4v_2 - 3v_3 + 4v_4\right) = \begin{pmatrix}0\\0\end{pmatrix} \quad -$$

$$\left(\frac{1}{4} + \frac{5}{48}\right)v_1 + \left(\frac{1}{6} - \frac{4}{48}\right)v_2 + \left(\frac{1}{2} + \frac{3}{48}\right)v_3 + \underbrace{\left(\frac{1}{12} - \frac{4}{48}\right)}_{0}v_4 = \begin{bmatrix}10/3\\5/2\end{bmatrix}$$

$$\boxed{\frac{17}{48}v_1 + \frac{4}{48}v_2 + \frac{27}{48}v_3 = \begin{bmatrix}10/3\\5/2\end{bmatrix}}$$

is a convex combination of 3 points of S

— the end —

Practice Problems

1. Let $\mathbf{v}_1 = \begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, and $\mathbf{p}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Determine whether $\mathbf{p}_1$ and $\mathbf{p}_2$ are in conv S.
2. Let S be the set of points on the curve $y = 1/x$ for $x > 0$. Explain geometrically why conv S consists of all points on and above the curve S.

Solutions to Practice Problems

1. The points $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are not orthogonal, so compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \ \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -8 \\ 2 \\ -3 \end{bmatrix}, \ \mathbf{p}_1 - \mathbf{v}_1 = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \mathbf{p}_2 - \mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}$$

Augment the matrix $[\, \mathbf{v}_2 - \mathbf{v}_1 \quad \mathbf{v}_3 - \mathbf{v}_1 \,]$ with both $\mathbf{p}_1 - \mathbf{v}_1$ and $\mathbf{p}_2 - \mathbf{v}_1$, and row reduce:

$$\begin{bmatrix} 1 & -8 & -5 & -3 \\ -1 & 2 & 1 & 0 \\ 3 & -3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 1 & \frac{2}{3} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{5}{2} \end{bmatrix}$$

The third column shows that $\mathbf{p}_1 - \mathbf{v}_1 = \frac{1}{3}(\mathbf{v}_2 - \mathbf{v}_1) + \frac{2}{3}(\mathbf{v}_3 - \mathbf{v}_1)$, which leads to $\mathbf{p}_1 = 0\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3$. Thus $\mathbf{p}_1$ is in conv S. In fact, $\mathbf{p}_1$ is in conv $\{\mathbf{v}_2, \mathbf{v}_3\}$.

The last column of the matrix shows that $\mathbf{p}_2 - \mathbf{v}_1$ is not a linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$. Thus $\mathbf{p}_2$ is not an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, so $\mathbf{p}_2$ cannot possibly be in conv S.

An alternative method of solution is to row reduce the augmented matrix of homogeneous forms:

$$[\, \tilde{\mathbf{v}}_1 \quad \tilde{\mathbf{v}}_2 \quad \tilde{\mathbf{v}}_3 \quad \tilde{\mathbf{p}}_1 \quad \tilde{\mathbf{p}}_2 \,] \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. If $\mathbf{p}$ is a point above S, then the line through $\mathbf{p}$ with slope -1 will intersect S at two points before it reaches the positive x- and y-axes.