Sec 8.2 Affine independence

Def A set of points $\vec{v}_1, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ is called affinely dependent if there exist $c_1, \ldots, c_p$ in $\mathbb{R}$ (not all zero) such that

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}$$

← note: up to here this is the def of linearly dependent

and

$$c_1 + c_2 + \cdots + c_p = 0.$$

If no such numbers exist, we say the set is affinely independent

Note: An affine combination is a special type of linear combination (every affine combination is also a linear combination)

An affine dependence is a special type of linear dependence (every affinely dependent set is also linearly dependent)

THEOREM 5

Given an indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$, with $p \geq 2$, the following statements are logically equivalent. That is, either they are all true statements or they are all false.

a. S is affinely dependent.

b. One of the points in S is an affine combination of the other points in S.

c. The set $\{\mathbf{v}_2 - \mathbf{v}_1, \ldots, \mathbf{v}_p - \mathbf{v}_1\}$ in $\mathbb{R}^n$ is linearly dependent.

d. The set $\{\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_p\}$ of homogeneous forms in $\mathbb{R}^{n+1}$ is linearly dependent.

When S has three points:

The affine hull of two distinct points $\vec{p}, \vec{q}$ is the line through $\vec{p}$ and $\vec{q}$ (See discussion in Sec 8.1).

Let $\vec{r}$ be a third point on this line,

let $\vec{s}$ be another point not on this line.

Then $\{\vec{p}, \vec{q}, \vec{r}\}$ is affinely dependent.

Why? $\vec{r}$ being in aff$\{\vec{p}, \vec{q}\}$ means $\vec{r}$ is an affine combination of $\vec{p}$ and $\vec{q}$, so by part (b) of Thm 5 $\{\vec{p}, \vec{q}, \vec{r}\}$ is affinely dependent

In contrast, $\{\vec{p}, \vec{q}, \vec{s}\}$ is affinely independent.

Why? These three points are not collinear, (all lying on the same line)

meaning $\vec{s}$ is not in aff$\{\vec{p}, \vec{q}\}$,

$\vec{p}$ — " — aff$\{\vec{s}, \vec{q}\}$,

$\vec{q}$ — " — aff$\{\vec{s}, \vec{p}\}$.

For example, here's a set S of three points:

EXAMPLE 2 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 6.5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$, and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Determine whether S is affinely independent.

SOLUTION Compute $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ -.5 \end{bmatrix}$ and $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. These two points are not multiples and hence form a linearly independent set, S'. So all statements in Theorem 5 are false, and S is affinely independent. Figure 2 shows S and the translated set S'. Notice that Span S' is a plane through the origin and aff S is a parallel plane through $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. (Only a portion of each plane is shown here, of course.) ■

→ meaning $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are not all on the same line

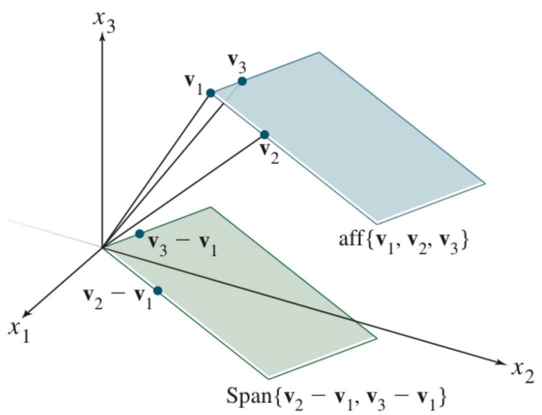


FIGURE 2 An affinely independent set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(Same as before)

EXAMPLE 3 Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \\ 6.5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 14 \\ 6 \end{bmatrix}$, and let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$. Is S affinely dependent?

If so, produce an affine dependence relation

$$c_1 \vec{v}_1 + \ldots + c_4 \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(where $c_1 + \cdots + c_4 = 0$ and not all c_i are zero)

Sol 1 We'll use part (d) of Thm 5.

Consider the homogeneous forms $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4$

$$M = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 7 & 4 & 14 \\ 7 & 6.5 & 7 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 3 pivot points and 4 columns,

so $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4\}$ is linearly dependent.

By Thm 5 (d), S is affinely dependent.

How to find an affine dependence relation?

First find a linear dependence relation

$$c_1 \tilde{v}_1 + c_2 \tilde{v}_2 + c_3 \tilde{v}_3 + c_4 \tilde{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad M \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then

$$\left.\begin{matrix} c_1 - 4c_4 = 0 \\ c_2 + 2c_4 = 0 \\ c_3 + 3c_4 = 0 \end{matrix}\right\} \quad \begin{matrix} c_1 = 4c_4 \\ c_2 = -2c_4 \\ c_3 = -3c_4 \end{matrix} \qquad c_4 \text{ is free}$$

A possible solution is $c_4 = 1$, $c_1 = 4$, $c_2 = -2$, $c_3 = -3$

So
$$4\begin{bmatrix}1\\3\\7\\1\end{bmatrix}-2\begin{bmatrix}2\\7\\6.5\\1\end{bmatrix}-3\begin{bmatrix}0\\4\\7\\1\end{bmatrix}+1\begin{bmatrix}0\\14\\6\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

But note the bottom entries are all 1,

so whatever $c_1, \ldots, c_4$ we choose would give $c_1 + \cdots + c_4 = 0$

(Check: $4-2-3+1=0$)

So $4\begin{bmatrix}1\\3\\7\end{bmatrix}-2\begin{bmatrix}2\\7\\6.5\end{bmatrix}-3\begin{bmatrix}0\\4\\7\end{bmatrix}+1\begin{bmatrix}0\\14\\6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ is an affine dependence relation for S

Alternative sol 2 (arriving at the same answer):

EXAMPLE 3 Let $\mathbf{v}_1 = \begin{bmatrix}1\\3\\7\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}2\\7\\6.5\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}0\\4\\7\end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix}0\\14\\6\end{bmatrix}$, and let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$. Is S affinely dependent?

SOLUTION Compute $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix}1\\4\\-.5\end{bmatrix}$, $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$, and $\mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix}-1\\11\\-1\end{bmatrix}$,

and row reduce the matrix:

$$\begin{bmatrix}1 & -1 & -1\\ 4 & 1 & 11\\ -.5 & 0 & -1\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -1\\ 0 & 5 & 15\\ 0 & -.5 & -1.5\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -1\\ 0 & 5 & 15\\ 0 & 0 & 0\end{bmatrix}$$

Recall from Section 4.5 (or Section 2.8) that the columns are linearly dependent because not every column is a pivot column; so $\mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{v}_4 - \mathbf{v}_1$ are linearly dependent. By statement (c) in Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely dependent. This dependence can also be established using (d) in Theorem 5 instead of (c). ■

The calculations in Example 3 show that $\mathbf{v}_4 - \mathbf{v}_1$ is a linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$, which means that $\mathbf{v}_4 - \mathbf{v}_1$ is in Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$. By Theorem 1 in Section 8.1, $\mathbf{v}_4$ is in aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In fact, complete row reduction of the matrix in Example 3 would show that

$$\mathbf{v}_4 - \mathbf{v}_1 = 2(\mathbf{v}_2 - \mathbf{v}_1) + 3(\mathbf{v}_3 - \mathbf{v}_1) \tag{5}$$
$$\mathbf{v}_4 = -4\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 \tag{6}$$

↑ ↑ ↑

These coefficients are called affine coordinates (or barycentric coordinates of $\vec{v}_4$)

Barycentric Coordinates

(Recall)

Thm 8 (Unique Representation Theorem for basis) in Sec 4.4

Let $\mathcal{B} = \{b_1, b_2, \dots, b_k\}$ be a linearly independent set.

Then every element in $V = \text{Span}\,\mathcal{B}$ can be written as a linear combination of $\mathcal{B}$ in exactly one way. I.e,

for every element x in V, there exist a unique set of scalars $c_1, c_2, \cdots, c_k$ such that

$$x = c_1 b_1 + c_2 b_2 + \dots + c_n b_k.$$

This allowed us to define coordinates relative to a basis $\mathcal{B}$.

Now we have

Thm 6 (Unique Representation Theorem, affine version)

Let $S = \{b_1, b_2, \dots, b_k\}$ be an affinely independent set.

Then every element in aff S can be written as an affine combination of S in exactly one way. I.e,

for every point $\vec{x}$ in aff S, there exist a unique set of scalars $c_1, c_2, \cdots, c_k$ such that

$$\vec{x} = c_1 b_1 + c_2 b_2 + \dots + c_k b_k \quad \text{and} \quad c_1 + c_2 + \dots + c_k = 1$$

Def The coefficients in this unique representation of $\vec{x}$ are called the barycentric coordinates of $\vec{x}$

As I demonstrated in previous example...

Observe that (7) is equivalent to the single equation

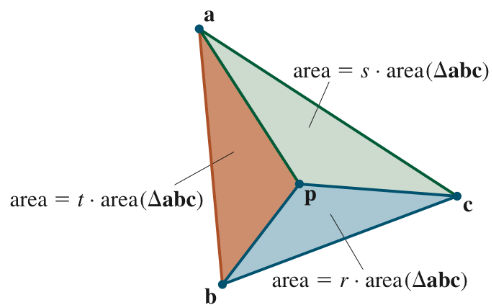
$$\begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + \cdots + c_k \begin{bmatrix} \mathbf{v}_k \\ 1 \end{bmatrix} \tag{8}$$

involving the homogeneous forms of the points. Row reduction of the augmented matrix $\begin{bmatrix} \tilde{\mathbf{v}}_1 & \cdots & \tilde{\mathbf{v}}_k & \tilde{\mathbf{p}} \end{bmatrix}$ for (8) produces the barycentric coordinates of $\mathbf{p}$.

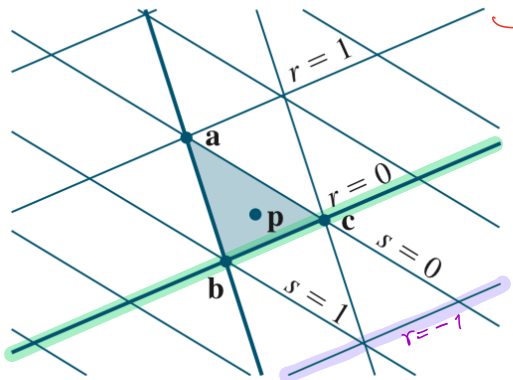
Geometric interpretation of barycentric coordinates:

If $\vec{p}$ is inside the triangle $\Delta \vec{a}\,\vec{b}\,\vec{c}$

then $\vec{p} = r\vec{a} + s\vec{b} + t\vec{c}$ where $r+s+t=1$ and r, s, t positive



When $\vec{p}$ is not inside the triangle, some of the barycentric coordinates will be negative.



aff$\{\vec{b}, \vec{c}\} + \vec{a}$

The points on this line is an affine combination of $\vec{b}$ and $\vec{c}$ only, so $r = 0$

FIGURE 5 Barycentric coordinates for points in aff $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

(Extra Ex)

EXAMPLE 4 Let $\mathbf{a} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find the barycentric coordinates of $\mathbf{p}$ determined by the affinely independent set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

SOLUTION Row reduce the augmented matrix of points in homogeneous form, moving the last row of ones to the top to simplify the arithmetic:

$$\begin{bmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} & \tilde{\mathbf{c}} & \tilde{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 9 & 5 \\ 7 & 0 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 5 \\ 7 & 0 & 3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{12} \end{bmatrix}$$

The coordinates are $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{5}{12}$, so $\mathbf{p} = \frac{1}{4}\mathbf{a} + \frac{1}{3}\mathbf{b} + \frac{5}{12}\mathbf{c}$. ■

Barycentric Coordinates in Computer Graphics

When working with geometric objects in a computer graphics program, a designer may use a "wire-frame" approximation to an object at certain key points in the process of creating a realistic final image. For instance, if the surface of part of an object consists of small flat triangular surfaces, then a graphics program can easily add color, lighting, and shading to each small surface when that information is known only at the vertices. Barycentric coordinates provide the tool for smoothly interpolating the vertex information over the interior of a triangle. The interpolation at a point is simply the linear combination of the vertex values using the barycentric coordinates as weights.

Colors on a computer screen are often described by RGB coordinates. A triple (r, g, b) indicates the amount of each color—red, green, and blue—with the parameters varying from 0 to 1. For example, pure red is $(1, 0, 0)$, white is $(1, 1, 1)$, and black is $(0, 0, 0)$.

EXAMPLE 5 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 3 \\ 3 \\ 3.5 \end{bmatrix}$. The colors at the vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ of a triangle are magenta $(1, 0, 1)$, light magenta $(1, .4, 1)$, and purple $(.6, 0, 1)$, respectively. Find the interpolated color at $\mathbf{p}$. See Figure 6.

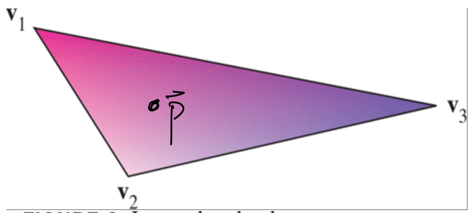


FIGURE 6 Interpolated colors.

SOLUTION First, find the barycentric coordinates of $\mathbf{p}$. Here is the calculation using homogeneous forms of the points, with the first step moving row 4 to row 1:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}} \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 4 & 1 & 3 \\ 1 & 3 & 5 & 3 \\ 5 & 4 & 1 & 3.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & .25 \\ 0 & 1 & 0 & .50 \\ 0 & 0 & 1 & .25 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathbf{p} = .25\mathbf{v}_1 + .5\mathbf{v}_2 + .25\mathbf{v}_3$. Use the barycentric coordinates of $\mathbf{p}$ to make a linear combination of the color data. The RGB values for $\mathbf{p}$ are

$$.25 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + .50 \begin{bmatrix} 1 \\ .4 \\ 1 \end{bmatrix} + .25 \begin{bmatrix} .6 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .9 \\ .2 \\ 1 \end{bmatrix} \begin{matrix} \text{red} \\ \text{green} \\ \text{blue} \end{matrix}$$

■