Ch 8 Geometry of vector spaces

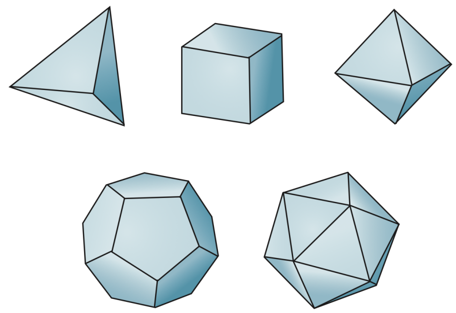


FIGURE 1 The five Platonic solids.

These are regular 3-D polytopes

Application:

linear programming (higher dim)

Computer graphics in $\mathbb{R}^3$

Sec 8.1 Affine combinations

Idea: Think of vectors in $\mathbb{R}^n$ as "points"

Subspaces of vector space $\mathbb{R}^3$ are

- $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$
- a line containing the origin
- a plane containing the origin
- $\mathbb{R}^3$

We will study lines & planes (that don't have to contain $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$), called flats

Def • An affine combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$

is a linear combination $c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$

such that $\underbrace{c_1 + c_2 + \ldots + c_p = 1}_{\text{the weights sum up to 1}}$

• The affine hull (or affine span) of a set S

is the set of all affine combinations of points in S.

(Notation: aff S)

<u>Ex 1</u> $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\quad \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Then $-5\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ is an affine combination

of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, so $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ is in $\text{aff}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

<u>Note</u> The affine hull of a single point $\vec{v}$, $\text{aff}\{\vec{v}\}$,
is just $\{\vec{v}\}$, since $c\,v$ with $c = 1$
is the only affine combination of $\vec{v}$

<u>Note</u> The affine hull of two distinct points $\vec{v}_1, \vec{v}_2$ is a <u>line</u>.

<u>Why</u>?

Suppose $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2$ is an affine combi of $\vec{v}_1, \vec{v}_2$.

Then $c_1 + c_2 = 1$. Let $c_2 = t$ any number in $\mathbb{R}$.

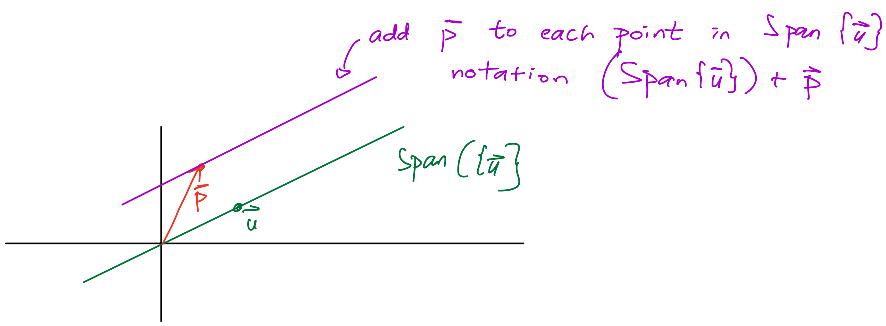
Then $c_1 = 1 - c_2 = 1 - t$.

Then $\text{aff}\{\vec{v}_1, \vec{v}_2\} = \{(1-t)\vec{v}_1 + t\vec{v}_2 \mid \text{any } t \text{ in } \mathbb{R}\}$

$= \{v_1 + t(\vec{v}_2 - \vec{v}_1) \mid \text{any } t \text{ in } \mathbb{R}\}$

This is the line that contains both $\vec{v}_1$ and $\vec{v}_2 - \vec{v}_1$

Alternatively, let $\vec{u} = \vec{v}_2 - \vec{v}_1$, let $\vec{p} = \vec{v}_1$



Theorem 1:

A point $\vec{y}$ in $\mathbb{R}^n$ is an affine combination of $\vec{v}_1$ & $\vec{v}_2$ in $\mathbb{R}^n$ iff $\vec{y}-\vec{v}_1$ is in Span$\{\vec{v}_2-\vec{v}_1\}$

In general:

A point $\vec{y}$ in $\mathbb{R}^n$ is an affine combination of $\vec{v}_1,\dots,\vec{v}_p$ in $\mathbb{R}^n$ iff $\vec{y}-\vec{v}_1$ is in Span$\{\vec{v}_2-\vec{v}_1, \vec{v}_3-\vec{v}_1, \dots, \vec{v}_p-\vec{v}_1\}$.

we can replace $\vec{v}_1$ with any of $\vec{v}_2,\dots,\vec{v}_p$.

<u>Exercise 2</u> Write $\vec{y}=\begin{bmatrix}5\\7\end{bmatrix}$ as an affine combination of the points

$$\vec{v}_1=\begin{bmatrix}1\\1\end{bmatrix} \quad \vec{v}_2=\begin{bmatrix}-1\\2\end{bmatrix} \quad \vec{v}_3=\begin{bmatrix}3\\2\end{bmatrix}$$

<u>Sol</u> Translate the points $\vec{v}_2, \vec{v}_3, \vec{y}$ by $-\vec{v}_1$:

$$\vec{v}_2-\vec{v}_1=\begin{bmatrix}-1\\2\end{bmatrix}-\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}, \quad \vec{v}_3-\vec{v}_1=\begin{bmatrix}3\\2\end{bmatrix}-\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}, \quad \vec{y}-\vec{v}_1=\begin{bmatrix}5\\7\end{bmatrix}-\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}4\\6\end{bmatrix}$$

We need to find scalars c_2, c_3 such that

$$c_2(\vec{v}_2-\vec{v}_1)+c_3(\vec{v}_3-\vec{v}_1)=\vec{y}-\vec{v}_1$$

To find c_2, c_3, row reduce the augmented matrix

$$\left[\begin{array}{cc|c}-2&2&4\\1&1&6\end{array}\right] \sim \left[\begin{array}{cc|c}+1&-1&-2\\1&1&6\end{array}\right] \sim \left[\begin{array}{cc|c}1&-1&-2\\0&2&8\end{array}\right] \sim \left[\begin{array}{cc|c}1&-1&-2\\0&1&4\end{array}\right] \sim \left[\begin{array}{cc|c}1&0&2\\0&1&4\end{array}\right] \quad \begin{matrix}c_2=2\\c_3=4\end{matrix}$$

$$\vec{y}-\vec{v}_1 = 2(\vec{v}_2-\vec{v}_1)+4(\vec{v}_3-\vec{v}_1)$$

$$y = \vec{v}_1 + 2v_2 - 2v_1 + 4\vec{v}_3 - 4\vec{v}_1$$

$$\boxed{y = -5\vec{v}_1 + 2v_2 + 4\vec{v}_3}$$

Check: $\begin{bmatrix}5\\7\end{bmatrix} \stackrel{?}{=} -5\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}-1\\2\end{bmatrix} + 4\begin{bmatrix}3\\2\end{bmatrix}$

$-5-2+12 = -5+10 = 5$ ✓

$-5+4+8 = -1+8 = 7$ ✓

Check: $1 \stackrel{?}{=} -5+2+4$ ✓

Main Def A set S is affine if:

If $\vec{p}, \vec{q}$ in S then $\underbrace{(1-t)\vec{p} + t\vec{q}}_{\text{the set of all such expressions is a line}} \in S$ for all t in $\mathbb{R}$

Geometric way to define an affine set:

A set S is affine if:

If two points $\vec{p}, \vec{q}$ are in S, then the line through $\vec{p}$ and $\vec{q}$ are in S

Main Thm: S is affine iff $S = \text{aff } S$.

I.e. S is affine iff every affine combination of S lies in S.

Def Let $S \subset \mathbb{R}^n$, let $\vec{p}$ be a vector in $\mathbb{R}^n$.

$S + \vec{p} = \{\vec{s} + \vec{p} \mid \vec{s} \text{ in } S\}$, called the translate of S by $\vec{p}$

Def
1. A flat in $\mathbb{R}^n$ is a translate of a subspace of $\mathbb{R}^n$

   ex: Any line and any plane is a flat in $\mathbb{R}^3$
2. Two flats are parallel if one is a translate of the orther.
3. The dimension of a flat is the dimension of the corresponding parallel subspace.
4. The dimension of a set S is the dimension of the smallest flat containing S.
5. A line in $\mathbb{R}^n$ is a flat of dimension 1.

Thm 3 Any nonempty set S is affine iff S is a flat.

So a flat consists of all the affine combinations of points in the set.

EXAMPLE 2 Let $\mathbf{b}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. The set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{R}^3$. Determine whether the points $\mathbf{p}_1$ and $\mathbf{p}_2$ are affine combinations of the points in $\mathcal{B}$.

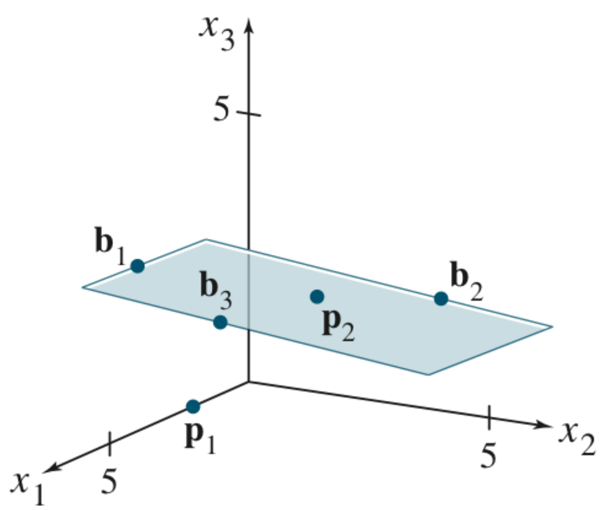
SOLUTION Find the $\mathcal{B}$-coordinates of $\mathbf{p}_1$ and $\mathbf{p}_2$. These two calculations can be combined by row reducing the matrix $[\,\mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \;\; \mathbf{p}_1 \;\; \mathbf{p}_2\,]$, with two augmented columns:

$$\begin{bmatrix} 4 & 0 & 5 & 2 & 1 \\ 0 & 4 & 2 & 0 & 2 \\ 3 & 2 & 4 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & \frac{2}{3} \\ 0 & 1 & 0 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & 2 & -\frac{1}{3} \end{bmatrix}$$

Read column 4 to build $\mathbf{p}_1$, and read column 5 to build $\mathbf{p}_2$:

$$\mathbf{p}_1 = -2\mathbf{b}_1 - \mathbf{b}_2 + 2\mathbf{b}_3 \quad \text{and} \quad \mathbf{p}_2 = \tfrac{2}{3}\mathbf{b}_1 + \tfrac{2}{3}\mathbf{b}_2 - \tfrac{1}{3}\mathbf{b}_3$$

The sum of the weights in the linear combination for $\mathbf{p}_1$ is -1, not 1, so $\mathbf{p}_1$ is *not* an affine combination of the $\mathbf{b}$'s. However, $\mathbf{p}_2$ *is* an affine combination of the $\mathbf{b}$'s, because the sum of the weights for $\mathbf{p}_2$ is 1. ■



Earlier, Theorem 1 displayed an important connection between affine combinations and linear combinations. The next theorem provides another view of affine combinations, which for $\mathbb{R}^2$ and $\mathbb{R}^3$ is closely connected to applications in computer graphics, discussed in the next section (and in Section 2.7).

DEFINITION For $\mathbf{v}$ in $\mathbb{R}^n$, the standard **homogeneous form** of $\mathbf{v}$ is the point $\tilde{\mathbf{v}} = \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}$ in $\mathbb{R}^{n+1}$.

THEOREM 4 A point $\mathbf{y}$ in $\mathbb{R}^n$ is an affine combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ in $\mathbb{R}^n$ if and only if the homogeneous form of $\mathbf{y}$ is in Span $\{\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_p\}$. In fact, $\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, with $c_1 + \cdots + c_p = 1$, if and only if $\tilde{\mathbf{y}} = c_1\tilde{\mathbf{v}}_1 + \cdots + c_p\tilde{\mathbf{v}}_p$.

—end of notes—

PROOF A point $\mathbf{y}$ is in aff $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ if and only if there exist weights $c_1, \dots, c_p$ such that

$$\begin{bmatrix} \mathbf{y} \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} \mathbf{v}_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \mathbf{v}_2 \\ 1 \end{bmatrix} + \cdots + c_p \begin{bmatrix} \mathbf{v}_p \\ 1 \end{bmatrix}$$

This happens if and only if $\tilde{\mathbf{y}}$ is in Span $\{\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \dots, \tilde{\mathbf{v}}_p\}$. ■

EXAMPLE 4 Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$. Use Theorem 4 to write $\mathbf{p}$ as an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, if possible.

SOLUTION Row reduce the augmented matrix for the equation

$$x_1\tilde{\mathbf{v}}_1 + x_2\tilde{\mathbf{v}}_2 + x_3\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}$$

To simplify the arithmetic, move the fourth row of 1's to the top (equivalent to three row interchanges). After this, the number of arithmetic operations here is basically the same as the number needed for the method using Theorem 1.

$$[\,\tilde{\mathbf{v}}_1 \;\; \tilde{\mathbf{v}}_2 \;\; \tilde{\mathbf{v}}_3 \;\; \tilde{\mathbf{p}}\,] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 4 \\ 1 & 2 & 7 & 3 \\ 1 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 1 \\ 0 & 1 & 6 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 1.5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & .5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 4, $1.5\mathbf{v}_1 - \mathbf{v}_2 + .5\mathbf{v}_3 = \mathbf{p}$. See Figure 4, which shows the plane that contains $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{p}$ (together with points on the coordinate axes). ■

Additional examples:

The next example takes a fresh look at a familiar set—the set of all solutions of a
2 system $A\mathbf{x} = \mathbf{b}$.

EXAMPLE 3 Suppose that the solutions of an equation $A\mathbf{x} = \mathbf{b}$ are all of the form $\mathbf{x} = x_3\mathbf{u} + \mathbf{p}$, where $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}$. Recall from Section 1.5 that this set is parallel to the solution set of $A\mathbf{x} = \mathbf{0}$, which consists of all points of the form $x_3\mathbf{u}$. Find points $\mathbf{v}_1$ and $\mathbf{v}_2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is aff $\{\mathbf{v}_1, \mathbf{v}_2\}$.

SOLUTION The solution set is a line through $\mathbf{p}$ in the direction of $\mathbf{u}$, as in Figure 1. Since aff $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a line through $\mathbf{v}_1$ and $\mathbf{v}_2$, identify two points on the line $\mathbf{x} = x_3\mathbf{u} + \mathbf{p}$. Two simple choices appear when $x_3 = 0$ and $x_3 = 1$. That is, take $\mathbf{v}_1 = \mathbf{p}$ and $\mathbf{v}_2 = \mathbf{u} + \mathbf{p}$, so that

$$\mathbf{v}_2 = \mathbf{u} + \mathbf{p} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -2 \end{bmatrix}$$

In this case, the solution set is described as the set of all affine combinations of the form

$$\mathbf{x} = (1 - x_3)\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} + x_3\begin{bmatrix} 6 \\ -3 \\ -2 \end{bmatrix}$$ ■

Practice Problem

Plot the points $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{p} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ on graph paper, and explain why $\mathbf{p}$ *must* be an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then find the affine combination for $\mathbf{p}$. [*Hint:* What is the dimension of aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?]

Solution to Practice Problem

Since the points $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are not collinear (that is, not on a single line), aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ cannot be one-dimensional. Thus, aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must equal $\mathbb{R}^2$. To find the actual weights used to express $\mathbf{p}$ as an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, first compute

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{p} - \mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

To write $\mathbf{p} - \mathbf{v}_1$ as a linear combination of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$, row reduce the matrix having these points as columns:

$$\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 2 \end{bmatrix}$$

Thus $\mathbf{p} - \mathbf{v}_1 = \frac{1}{2}(\mathbf{v}_2 - \mathbf{v}_1) + 2(\mathbf{v}_3 - \mathbf{v}_1)$, which shows that

$$\mathbf{p} = \left(1 - \tfrac{1}{2} - 2\right)\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2 + 2\mathbf{v}_3 = -\tfrac{3}{2}\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2 + 2\mathbf{v}_3$$

This expresses $\mathbf{p}$ as an affine combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, because the coefficients sum to 1.

Exercise 5

Let $\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\vec{b}_3 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$, and $S = \{ \vec{b}_1, \vec{b}_2, \vec{b}_3 \}$

Note that S is an orthogonal basis for $\mathbb{R}^3$.

Write $\vec{y} = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}$ as an affine combination of the points in S, if possible. [Hint: Use Theorem 5 in Sec 6.2 instead of row reduction to find the weights.]

Sol Thm 5 in Sec 6.2 says ...

$\boxed{\text{Let } \{\vec{b}_1, \dots, \vec{b}_n\} \text{ be an orthogonal basis for } \mathbb{R}^n. \text{ For each } \vec{y} \text{ in } \mathbb{R}^n, \text{ the weights in the linear combination } \vec{y} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n \text{ are given by } c_j = \frac{\vec{y} \cdot \vec{b}_j}{\vec{b}_j \cdot \vec{b}_j} \ (\text{for } j = 1, \dots, n)}$

$$c_1 = \frac{\vec{y} \cdot \vec{b}_1}{\vec{b}_1 \cdot \vec{b}_1} = \frac{\begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} = \frac{6+8+4}{4+1+1} = \frac{18}{6} = 3 \checkmark$$

$c_2 = -1$

$c_3 = -1$

So $\vec{y} = 3\vec{b}_1 - \vec{b}_2 - \vec{b}_3$.

Since the weights $c_1 + c_2 + c_3 = 1$,

$\boxed{\vec{y} \text{ is an affine combination of } S}$.