# Sec 7.4 The singular value decomposition (SVD)

Goal: Not all matrices can be factored as $A = P\underbrace{D}_{\text{diagonal}}P^{-1}$,

but every m×n matrix can be factored as $A = Q\Sigma P^{-1}$.

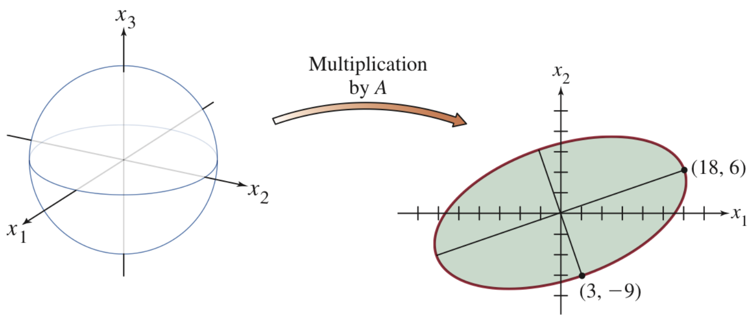
Recall (Ch 5) if $\lambda_1$ is an eigenvalue of n×n $A$ w/ the greatest length, then a corresponding unit eigenvector $v_1$ gives the direction in which the stretching effect of $A$ is greatest:

$$\|A\vec{v}_1\| = \|\lambda_1 \vec{v}_1\|$$

We can generalize this concept to m×n matrices, & this will lead to the SVD.

optional intro

Ex 1: $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.




The linear map $\vec{x} \longmapsto A\vec{x}$ maps the unit sphere in $\mathbb{R}^3$ onto an ellipse in $\mathbb{R}^2$.

Problem: find a unit vector at which the length $\|A\vec{x}\|$ is maximized & compute $\|A\vec{x}\|$.

Sol:

Note:
- $\|A\vec{x}\|^2$ is maximized at the same $\vec{x}$ that maximizes $\|A\vec{x}\|$.
- $\|A\vec{x}\|^2 = A\vec{x}\cdot A\vec{x} = (A\vec{x})^T(A\vec{x}) = \vec{x}^TA^T(A\vec{x}) = \vec{x}^T(A^TA)\vec{x}$

  def of inner product
- $A^TA$ is a symmetric matrix, since $(A^TA)^T = A^T(A^T)^T = A^TA$

This page is part of optional intro

Fact Let $B$ be a symmetric matrix.

(Thm 6 in Sec 7.3) Let $M$ be the largest possible value $\vec{x}^T B \vec{x}$ for unit vectors $\vec{x}$. (Note: $M$ is a real number)

Then: ① $M$ = the largest eigenvalue $\lambda$ of $B$

② If $\vec{v}$ is a unit eigenvector of $\lambda$, then $\vec{v}^T B \vec{v} = M$.

- So the maximum value $M$ of $\|A\vec{x}\|^2$ is equal to the greatest eigenvalue $\lambda$ of $B = A^T A$. And the maximum value is attained at a unit eigenvector of $B = A^T A$ corresponding to $\lambda$.

For this problem, compute

$$A^T A = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

The eigenvalues of $A^T A$ are $\lambda_1 = 360$, $\lambda_2 = 90$, and $\lambda_3 = 0$. Corresponding unit eigenvectors are, respectively,

$$\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

The maximum value of $\|A\mathbf{x}\|^2$ is 360, attained when $\mathbf{x}$ is the unit vector $\mathbf{v}_1$. The vector $A\mathbf{v}_1$ is a point on the ellipse in Figure 1 farthest from the origin, namely

$$A\mathbf{v}_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

For $\|\mathbf{x}\| = 1$, the maximum value of $\|A\mathbf{x}\|$ is $\|A\mathbf{v}_1\| = \sqrt{360} = 6\sqrt{10}$. ■

# Beginning of lecture about SVD

## The singular values of an $m \times n$ matrix

Let $A$ be an $m \times n$ matrix. Then $A^T A$ is symmetric, and so, by Sec 7.1, can be orthogonally diagonalized.

Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$, and let $\lambda_1, \ldots, \lambda_n$ be the associated eigenvalues of $A^T A$.

Then, for each eigenvector $\vec{v}_i$,

$$\|A\vec{v}_i\|^2 \underset{\text{def of inner product}}{=} (A\vec{v}_i)^T (A\vec{v}_i) = \vec{v}_i^T \underbrace{A^T A \vec{v}_i}$$

$$= \vec{v}_i^T \; \lambda_i \vec{v}_i \qquad \text{since } \vec{v}_i \text{ is a } \lambda_i\text{-eigenvector of } A^T A$$

$$= \lambda_i \; \vec{v}_i^T \vec{v}_i$$

$$= \lambda_i \underbrace{\|\vec{v}_i\|^2}$$

$$= \lambda_i \; (1) \qquad \text{since } \vec{v}_i \text{ is a unit vector}$$

Thus $\|A\vec{v}_i\| \overset{(*)}{=} \sqrt{\lambda_i}$

The **singular values** of $A$ are the square roots of the eigenvalues of $A^T A$, denoted by $\sigma_1, \ldots, \sigma_n$, and they are arranged in decreasing order. That is, $\sigma_i = \sqrt{\lambda_i}$ for $1 \leq i \leq n$. By equation (*), *the singular values of A are the lengths of the vectors* $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$.

Ex Find the singular values of $A = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$

Sol: • $A^T A = \begin{bmatrix} 7 & 0 & 4 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 65 & 30 \\ 30 & 20 \end{bmatrix}$

• Find the eigenvalues of $A^TA$:

$$\det(A^TA - \lambda I) = \det\begin{pmatrix} 65-\lambda & 30 \\ 30 & 20-\lambda \end{pmatrix}$$

$$= (65-\lambda)(20-\lambda) - 30^2$$

$$= \lambda^2 - 85\lambda + 1300 - 900$$

$$= \lambda^2 - 85\lambda + 400$$

$$= (\lambda - 80)(\lambda - 5)$$

Eigenvalues of $A^TA$ are $\lambda_1 = 80$, $\lambda_2 = 5$

• The singular values are $\sigma_1 > \sigma_2$

$$4\sqrt{5} = \sqrt{80} > \sqrt{5}$$

Note: If $v_1$ and $v_2$ are unit eigenvectors corresponding to $\lambda_1, \lambda_2$,

then

$$\|A\vec{v}_i\| \overset{(*)}{=} \sqrt{\lambda_i} = \sigma_i$$

(Thm 9) Suppose $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $A^TA$, arranged so that the corresponding eigenvalues of $A^TA$ satisfy

$$\lambda_1 \geqslant \cdots \geqslant \lambda_n.$$

Let $r$ = # of nonzero singular values of $A$.

Then: • rank $A = r$

• $\{A\vec{v}_1, \dots, A\vec{v}_r\}$ is an orthogonal basis for Col $A$.

## Numerical Notes

In some cases, the rank of $A$ may be very sensitive to small changes in the entries of $A$. The obvious method of counting the number of pivot columns in $A$ does not work well if $A$ is row reduced by a computer. Roundoff error often creates an echelon form with full rank.

In practice, the most reliable way to estimate the rank of a large matrix $A$ is to count the number of nonzero singular values. In this case, extremely small nonzero singular values are assumed to be zero for all practical purposes, and the *effective rank* of the matrix is the number obtained by counting the remaining nonzero singular values.[1]

## The Singular Value Decomposition

The decomposition of $A$ involves an $m \times n$ "diagonal" matrix $\Sigma$ of the form

$$\begin{array}{cl} \Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} & \leftarrow m - r \text{ rows} \\ \qquad\qquad\quad \uparrow n - r \text{ columns} & \end{array}$$

Theorem 10 (The Singular value decomposition)

Let $A$ be an $m \times n$ matrix w/ rank $r$.

Then there exist :

(1) an $m \times n$ matrix $\Sigma = \begin{bmatrix} \begin{matrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 \ldots & \\ & & & \sigma_r \end{matrix} & 0 \\ 0 & 0 \end{bmatrix}$ where

$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant \cdots \geqslant \sigma_r > 0$ are the first $r$ singular values of $A$

(2) an $m \times m$ orthogonal matrix $U$

(3) an $n \times n$ orthogonal matrix $V$

such that

$$A = U \Sigma V^T$$

The columns of $U$ are called <u>left singular vectors</u> of $A$

The columns of $V$ are called <u>right singular vectors</u> of $A$

- Any such factorization is called a <u>singular value decomposition</u> of $A$.
- The diagonal entries of $\Sigma$ are necessarily the singular values of $A$ (so $\Sigma$ is unique)
- The matrices $U$ and $V$ are not uniquely determined by $A$.

How to find a singular value decomposition of $A$ (that is also a proof for Thm 10 (SVD)):

(1) Find the eigenvalues of $A^TA$ $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (in descending order)

Take square roots to get the nonzero singular values

$$\begin{array}{ccccc} \sqrt{\lambda_1} & \sqrt{\lambda_2} & & \sqrt{\lambda_r} \\ \| & & & \\ \sigma_1 & \geq \sigma_2 & \geq \cdots \geq & \sigma_r \end{array}$$

Let $D = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$, Let $\Sigma = \left[\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ & & & \sigma_r & \\ \hline & & 0 & & 0 \end{array}\right]$ Size $m \times n$ same size as $A$

(2) Find the eigenvectors of $A^TA$ associated to $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Normalize these eigenvectors to find unit eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

Let $V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$

(3) Compute $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_r$

Normalize them $\vec{u}_1 := \frac{1}{\sigma_1} A\vec{v}_1$, $\vec{u}_2 = A\vec{v}_2, \ldots, \vec{u}_r = A\vec{v}_r$.

since $\|A\vec{v}_1\|^2 = \lambda_1$ as we showed earlier in eq $(*)$

Complete the linearly independent set $\{\vec{u}_1, \ldots, \vec{u}_r\}$ to an orthogonal basis $\{\vec{u}_1, \ldots, \vec{u}_r, \ldots, \vec{u}_m\}$ for $\mathbb{R}^m$

Let $U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$

Note

$$U\Sigma = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_m\,] \left[\begin{array}{cccc|c} \sigma_1 & & & 0 & \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ 0 & & & \sigma_r & \\ \hline & & 0 & & 0 \end{array}\right]$$

$$= [\,\sigma_1\mathbf{u}_1 \quad \cdots \quad \sigma_r\mathbf{u}_r \quad \mathbf{0} \quad \cdots \quad \mathbf{0}\,]$$
$$= AV$$

So $\quad U\Sigma V^T = AVV^T$

$\quad = A \quad$ since $V$ is orthogonal

— end of algorithm (also proof) for SVD —

Ex of SVD

Let $A = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$ as before

(Step 1) $A^TA = \begin{bmatrix} 7 & 0 & 4 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 65 & 30 \\ 30 & 20 \end{bmatrix}$ has eigenvalues $\lambda_1 = 80$, $\lambda_2 = 5$

From prev ex

So the singular values of A are $\sqrt{\lambda_1} = \sqrt{80} = 4\sqrt{5}$, $\sqrt{\lambda_2} = \sqrt{5}$

Let $D = \begin{bmatrix} 4\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$, let $\Sigma = \begin{bmatrix} 4\sqrt{5} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{bmatrix}$ (note size of $\Sigma$ is $3 \times 2$)

(Step 2)

Find the eigenvectors of $A^TA$ associated to $\lambda_1 = 80$, $\lambda_2 = 5$

For $\lambda_1 = 80$: Find basis for $\mathrm{Nul}\,(A^TA - 80\,\mathrm{Id}) = \mathrm{Nul}\left(\begin{bmatrix} 65-80 & 30 \\ 30 & 20-80 \end{bmatrix}\right)$

$$\left[\begin{array}{cc|c} -15 & 30 & 0 \\ 30 & -60 & 0 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad x_1 - 2x_2 = 0$$

$x_2$ can be any nonzero number, so let $x_2 = 1$. Then $x_1 = 2$

An 80-eigenvector for $A^TA$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Normalize: $\vec{v_1} = \dfrac{1}{\sqrt{2^2+1}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

For $\lambda_2 = 5$: Find basis for $\mathrm{Nul}\,(A^TA - 5\,\mathrm{Id}) = \mathrm{Nul}\left(\begin{bmatrix} 65-5 & 30 \\ 30 & 20-5 \end{bmatrix}\right)$

$$\left[\begin{array}{cc|c} 60 & 30 & 0 \\ 30 & 15 & 0 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad 2x_1 + x_2 = 0 \qquad \begin{aligned} 2x_1 &= -x_2 \\ x_1 &= -\frac{x_2}{2} \end{aligned}$$

Let $x_2 = 2$. Then $x_1 = -1$.

So a 5-eigenvector for $A^TA$ is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Normalize: $\vec{v_2} = \dfrac{1}{\sqrt{1^2+2^2}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

Let $V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$, then $V^T = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

(Step 3) Compute $A\vec{v}_1$, $A\vec{v}_2$

$$A\vec{v}_1 = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 16/\sqrt{5} \\ 0 \\ 12/\sqrt{5} \end{bmatrix}$$

$$A\vec{v}_2 = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -3/\sqrt{5} \\ 0 \\ 4/\sqrt{5} \end{bmatrix}$$

Normalize: $\vec{u}_1 := \frac{1}{\sqrt{\lambda_1}} A\vec{v}_1 = \frac{1}{4\sqrt{5}} \begin{bmatrix} 16/\sqrt{5} \\ 0 \\ 12/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix}$

Since $\|A\vec{v}_1\| = \sqrt{\lambda_1}$

Normalize: $\vec{u}_2 = \frac{1}{\sqrt{\lambda_2}} A\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -3/\sqrt{5} \\ 0 \\ 4/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -3/5 \\ 0 \\ 4/5 \end{bmatrix}$

$S = \{\vec{u}_1, \vec{u}_2\}$ is a linearly independent set in $\mathbb{R}^3$.

Extend $S$ to a basis of $\mathbb{R}^3$ by finding $\vec{u}_3$ such that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is orthogonal.

We want $\vec{u}_1 \cdot \vec{x} = 0$ and $\vec{u}_2 \cdot \vec{x} = 0$

Set $\frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ and $\frac{1}{5}\begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

So $4x_1 + 3x_3 = 0$ and $-3x_1 + 4x_3 = 0$

$$\left[\begin{array}{ccc|c} 4 & 0 & 3 & 0 \\ -3 & 0 & 4 & 0 \end{array}\right] \xrightarrow{\text{Row reduce}} \left[\begin{array}{ccc|c} 12 & 0 & 9 & 0 \\ -12 & 0 & 16 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 12 & 0 & 9 & 0 \\ 0 & 0 & 25 & 0 \end{array}\right]$$

$x_2$ can be any number

$$\left.\begin{array}{r} 12x_1 + 9x_3 = 0 \\ 25x_3 = 0 \end{array}\right\} \begin{array}{l} x_1 = 0 \\ x_3 = 0 \end{array}$$

We can choose $\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, already a unit vector.

Alternatively, note that both $\vec{u}_1, \vec{u}_2$ have 0 in 2nd entry, so I could have guessed $\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ would work

$$\text{Let } U = \begin{bmatrix} 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \\ 3/5 & 4/5 & 0 \end{bmatrix}$$

$$U\Sigma V^T = \begin{bmatrix} 4/5 & -3/5 & 0 \\ 0 & 0 & 1 \\ 3/5 & 4/5 & 0 \end{bmatrix} \begin{bmatrix} 4\sqrt{5} & 0 \\ 0 & \sqrt{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

is a singular value decomposition of $A = \begin{bmatrix} 7 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}$

— end of example —

**— we ended here on Thurs —**

Bases for fundamental subspaces (see Example 6 in book)

Perform an SVD for an $m \times n$ matrix $A$. Let $r = \text{rank } A$.

1. $\{A\vec{v}_1, \ldots, A\vec{v}_r\}$ is an orthogonal basis for Col $A$ (Thm 9)

   So $\{$the first $r$ columns of $U\} = \{\vec{u}_1, \ldots, \vec{u}_r\}$

   is an orthonormal basis for $\underbrace{\text{col } A}$

   called image or range of $A$

2. In Sec 6.1, we said $(\text{Col } A)^\perp = \text{Nul}(A^T)$

   So $\{$the rest of the columns of $U\} = \{\vec{u}_{r+1}, \ldots, \vec{u}_m\}$

   is an orthonormal basis for $\underbrace{\text{Nul}(A^T)}$

   called cokernel of $A$

3. Since $\|A\vec{v}_i\| = \sigma_i$ for $1 \le i \le n$ and $\sigma_i = 0$ iff $i > r$,

   $\{\vec{v}_{r+1}, \ldots, \vec{v}_n\} = \{$the last $n-r$ columns of $V\}$

   is an orthonormal basis for $\underbrace{\text{Nul } A}$

   called the kernel of $A$

4. We also have $(\text{Nul } A)^\perp = \text{Row } A$

   So $\{\vec{v}_1, \ldots, \vec{v}_r\} = \{$the first $r$ columns of $V\}$

   is an orthonormal basis for Row $A$.

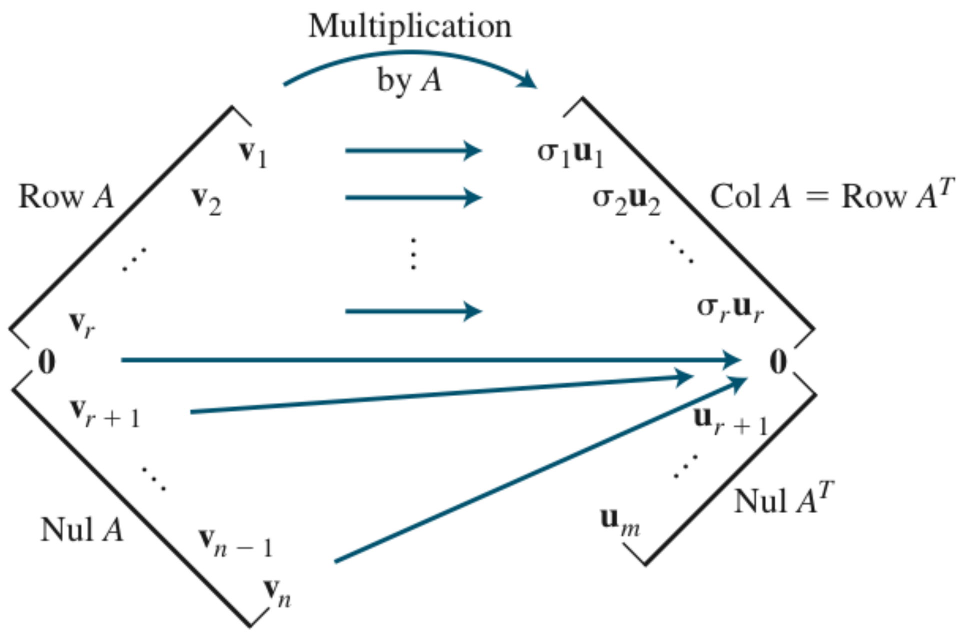



**FIGURE 4** The four fundamental subspaces and the action of $A$.

The Invertible matrix theorem (the end of the list)

Let $A$ be an $n \times n$ matrix. TFAE:

(a) $A$ is invertible.

(b) ⋮

(c) ⋮

(s) $(\text{Col } A)^{\perp} = \{\vec{0}\}$

(t) $(\text{Nul } A)^{\perp} = \mathbb{R}^n$

(u) Row $A = \mathbb{R}^n$

(v) $A$ has $n$ nonzero singular values

Additional Ex of SVD from book

**EXAMPLE 4** Find a singular value decomposition of $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

SOLUTION First, compute $A^TA = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$. The eigenvalues of $A^TA$ are 18 and 0, with corresponding unit eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

These unit vectors form the columns of $V$:

$$V = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

The singular values are $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and $\sigma_2 = 0$. Since there is only one nonzero singular value, the "matrix" $D$ may be written as a single number. That is, $D = 3\sqrt{2}$. The matrix $\Sigma$ is the same size as $A$, with $D$ in its upper left corner:

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

To construct $U$, first construct $A\mathbf{v}_1$ and $A\mathbf{v}_2$:

$$A\mathbf{v}_1 = \begin{bmatrix} 2/\sqrt{2} \\ -4/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix}, \quad A\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

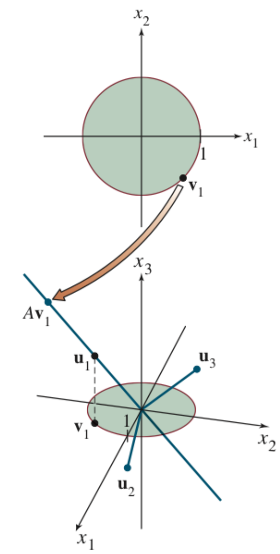


**FIGURE 3**

As a check on the calculations, verify that $\|A\mathbf{v}_1\| = \sigma_1 = 3\sqrt{2}$. Of course, $A\mathbf{v}_2 = \mathbf{0}$ because $\|A\mathbf{v}_2\| = \sigma_2 = 0$. The only column found for $U$ so far is

$$\mathbf{u}_1 = \frac{1}{3\sqrt{2}} A\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

The other columns of $U$ are found by extending the set $\{\mathbf{u}_1\}$ to an orthonormal basis for $\mathbb{R}^3$. In this case, we need two orthogonal unit vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ that are orthogonal to $\mathbf{u}_1$. (See Figure 3.) Each vector must satisfy $\mathbf{u}_1^T\mathbf{x} = 0$, which is equivalent to the equation $x_1 - 2x_2 + 2x_3 = 0$. A basis for the solution set of this equation is

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(Check that $\mathbf{w}_1$ and $\mathbf{w}_2$ are each orthogonal to $\mathbf{u}_1$.) Apply the Gram–Schmidt process (with normalizations) to $\{\mathbf{w}_1, \mathbf{w}_2\}$, and obtain

$$\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -2/\sqrt{45} \\ 4/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

Finally, set $U = [\,\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3\,]$, take $\Sigma$ and $V^T$ from above, and write

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

■

— end of PDF —