Sec 4.5 The dimension of a vector space

I. Def of dimension

Thm (Thm 10) If a vector space V has a basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ and S is any set in V containing more than n elements, then S must be linearly dependent.

Contrapositive statement (equivalent):

If S is any set in V that is linearly independent, then S contains no more than n elements.

Thm (Thm 11) If a vector space V has a basis of n elements, then every basis of V must consist of exactly n elements.

Def: If a vector space V has a finite spanning set, then V is said to be <u>finite-dimensional</u>, and <u>the dimension of V</u> is

$$\dim V = \text{the number of elements in a basis for } V$$

The dimension of the zero vector space $\{\text{zero element}\}$ is defined to be 0.

If V doesn't have a finite spanning set, then V is said to be <u>infinite-dimensional</u>.

<u>Ex</u> 1. $\dim \mathbb{R}^n = n$ because the standard basis for $\mathbb{R}^n$ is $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$

$\dim \mathbb{R}^3 = 3$ because $\mathbb{R}^3$ has a basis $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

2. $\dim \mathbb{P}_n = n+1$ because the standard basis for $\mathbb{P}_n$ is $\{1, t, t^2, \ldots, t^n\}$

$\dim \mathbb{P}_2 = 3$ because $\mathbb{P}_2$ has a basis $\{1, t, t^2\}$

so $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$ because Thm 9 (in Sec 4.4) says every vector space of dimension n is isomorphic to $\mathbb{R}^n$

3. $\mathbb{P} = \{\text{all polynomials}\}$ is infinite-dimensional

The space of all continuous functions is infinite-dimensional.

II. Subspaces of $\mathbb{R}^n$

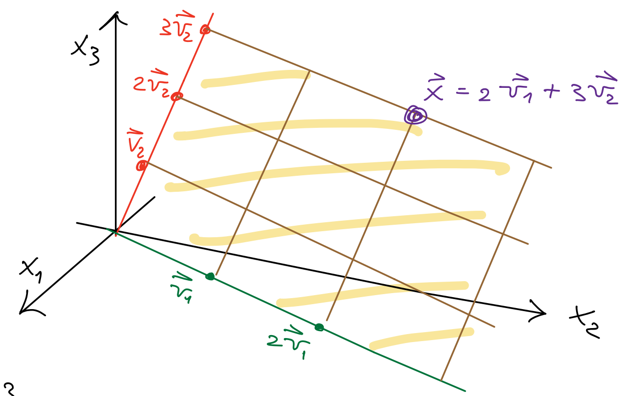
<u>Ex 7 from Sec 4.4 (previous lecture)</u>

The space $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix}3\\6\\2\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

has a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, so $\dim H = 2$

<u>Picture</u>

Basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ imposes a coordinate system ("graph paper") on the plane H in $\mathbb{R}^3$



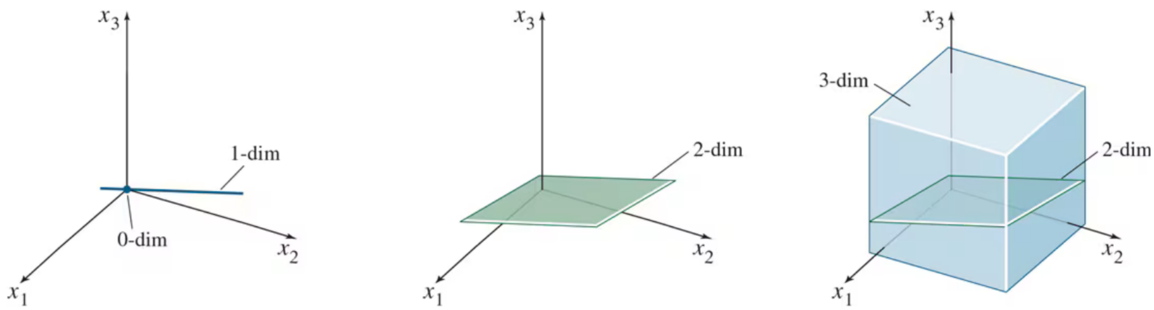
Note: • H is a subspace of $\mathbb{R}^3$

• H is isomorphic to the plane $\mathbb{R}^2$ (not $\mathbb{R}^3$) because Thm 9 (in Sec 4.4) says every vector space of dimension n is isomorphic to $\mathbb{R}^n$

<u>Fact</u> The following are all subspaces of $\mathbb{R}^3$ (classified by dimension):

- The only 0-dimensional subspace is the zero subspace $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$
- Every 1-dimensional subspace of $\mathbb{R}^3$ is a line L through the origin. L can be spanned by any nonzero vector contained in L.
- Every 2-dimensional subspace of $\mathbb{R}^3$ is a plane H through the origin. H can be spanned by any two linearly independent vectors in H
- The only 3-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself. $\mathbb{R}^3$ can be spanned by any three linearly independent vectors in $\mathbb{R}^3$, by the Invertible Matrix Theorem

Sample subspaces of $\mathbb{R}^3$



III. The basis theorem

Thm (Thm 12) Suppose V is a finite-dimensional vector space, and let H be a subspace of V.

(a) Any linearly independent set S in H (if S is not already a basis for H) can be expanded to a basis for H

(b) $\dim H \leq \dim V$

Remark: To be a basis for a vector space V, a set S must be:

- a spanning set of V, AND
- linearly independent (usually easier to verify than spanning)

BUT, if you know the dimension of V, and you know S has the correct number of elements, it's enough to show one of the two properties.

Thm (The Basis Theorem)
(Thm 13) Let V be a vector space with $\dim V = p \geq 1$.

(a) If S is a linearly independent set of p elements in V, then S is automatically a basis for V.

(b) If S is a set of p elements such that $V = \text{span}\{S\}$, then S is automatically a basis for V.

IV. The dimensions of Nul A & Col A (and Row A)

Def Let A be an $m \times n$ matrix.
rows cols

- The rank of A is dim Col A (notation: rank A)
- The nullity of A is dim Nul A (notation: nullity A)

Note rank A = number of pivot columns of A
= dim Row A

Why? • Our algorithm for finding a basis for Col A is to take $\mathcal{B} = \{$pivot columns of the original matrix $A\}$
(See Sec 4.3 pg 225-227)

• Our algorithm for finding a basis for Row A is to take $\mathcal{B} = \{$ nonzero rows of a row echelon form of A $\}$, and each nonzero row corresponds to a pivot position
(See Sec 4.3 pg 227)

Note
nullity A = number of free variables
= number of columns of A that are not pivot columns

Why? The algorithm for finding a basis for Nul A is to find a vector for each free variable.
(See Sec 4.2 pg 213-214)

Thm (The Rank Theorem)
(Thm 14) rank A + nullity A = number of columns in A

Proof. rank A + nullity A = (# of pivot columns) + (# of non-pivot columns) by above notes
= # of columns $\square$

Ex Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

Row reduce A to an echelon form:

Three nonpivot columns

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Two pivot columns

Then: 1) $\dim \operatorname{Col} A = 2 = \dim \operatorname{Row} A$

2) $\operatorname{rank} A = 2$

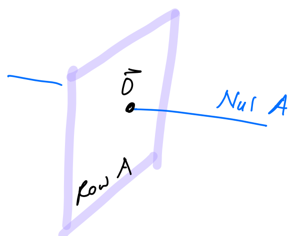
3) $\dim \operatorname{Nul} A = 3$

4) $\operatorname{Nullity} A = 3$

Check: $\underbrace{\operatorname{rank} A}_{2} + \underbrace{\operatorname{nullity} A}_{3} = \underbrace{\text{number of columns of } A}_{5}$

Looking ahead: We'll see later that $\operatorname{Row} A$ and $\operatorname{Nul} A$ are perpendicular to each other.

Cartoon in $\mathbb{R}^3$:



IV. The Invertible Matrix Theorem (Continued)

Let A be an $n \times n$ matrix. TFAE:

(1) A is invertible

(2) The columns of A form a basis of $\mathbb{R}^n$

(3) $\operatorname{Col} A = \mathbb{R}^n$

(4) $\operatorname{rank} A = n$

(5) $\operatorname{nullity} A = 0$ (the number zero)

(6) $\operatorname{Nul} A = \{\vec{0}\}$ (the zero vector space)

the zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^n$

Practice problems at the end of class

True or false? Give a reason (in 1-2 sentences)

Here V is a nonzero finite-dimensional vector space

(1) If $\dim V = 4$ and if S is a linearly dependent subset of V, then S contains more than 4 elements

Answer: False. Counter example: Let $V = \mathbb{R}^4$ and let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} \right\}$.

Here S is linearly dependent and S has no more than 4 elements.

(2) If S spans V and if T is a subset of V that contains more elements than S, then T is linearly dependent.

Answer: True. Since S is a spanning set for V, S contains a basis $\mathcal{B}$ for V (by "The Spanning Set Theorem" part (b) Thm 5 of Sec 4.3 pg 225). Since T has more elements than S, T also has more elements than $\mathcal{B}$, so (by Thm 10 of Sec 4.5 pg 241) T is linearly dependent.

(3)

Example 6

a. If A is a 7×9 matrix with nullity 2, what is the rank of A?

b. Could a 6×9 matrix have nullity 2?

Solution

a. Since A has 9 columns, $(\text{rank } A) + 2 = 9$, and hence rank $A = 7$.

b. No. If a 6×9 matrix, call it B, had a two-dimensional null space, it would have to have rank 7, by the Rank Theorem. But the columns of B are vectors in $\mathbb{R}^6$, and so the dimension of Col B cannot exceed 6; that is, rank B cannot exceed 6.

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