Sec 4.4 Coordinate systems

Idea : $\mathbb{R}^n$ w/ the standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ is nice

Put a "Coordinate system" on any vector space V

so that V acts like $\mathbb{R}^n$ w/ the standard basis.

I. Def of coordinates relative to a basis $\mathcal{B}$

Thm 8 (Unique Representation Theorem)

Let $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$ be a basis for a vector space V.

Then every element in V can be written as a linear combination of $\mathcal{B}$ in exactly one way. I.e,

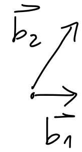
for every element x in V, there exist a unique set of scalars $c_1, c_2, \dots, c_n$ such that

$$x = c_1 b_1 + c_2 b_2 + \dots + c_n b_n.$$

Def • These weights $c_1, c_2, \dots, c_n$ are called the coordinates of x relative to the basis $\mathcal{B}$ or the $\mathcal{B}$-coordinates of x.

• The vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ in $\mathbb{R}^n$, denoted by $[x]_{\mathcal{B}}$,

is called the coordinate vector of x relative to $\mathcal{B}$ or the $\mathcal{B}$-coordinate vector of x.

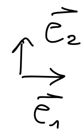


<u>Ex</u> $V = \mathbb{R}^2$, with basis $\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$.

Suppose $\vec{x}$ in V has coordinate vector (relative to $\mathcal{B}$) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\vec{x}$.

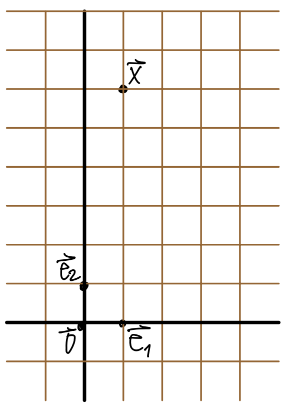
Ans: The $\mathcal{B}$-coordinates of $\vec{x}$ are -2 and 3, so

$$\vec{x} = -2\vec{b}_1 + 3\vec{b}_2 = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

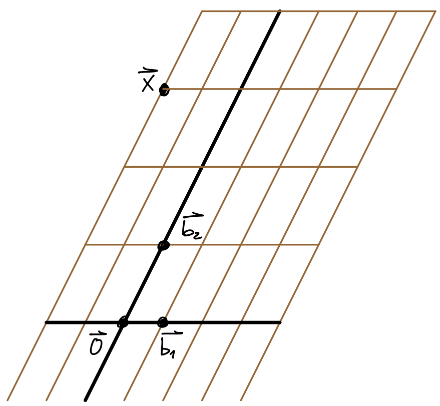


<u>Ex</u> $V = \mathbb{R}^2$, with standard basis $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$, $\vec{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

Then $\vec{x} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the coordinate vector of $\vec{x}$ relative to the standard basis is $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, the same as x itself



Standard graph paper for standar basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$



$\mathcal{B}$-graph paper for $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Ex $V = \mathbb{P}_2$, with basis $\mathcal{B} = \{\underbrace{1+t}_{b_1}, \underbrace{1+t^2}_{b_2}, \underbrace{t+t^2}_{b_3}\}$

Practice Problem #2 pg 238

Suppose p in V has coordinate vector (relative to $\mathcal{B}$)

$$[p]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}. \quad \text{Find } p$$

Ans: The $\mathcal{B}$-coordinates of p are $5, 1, -2$, so

$$p = 5\overbrace{(1+t)}^{b_1} + 1\overbrace{(1+t^2)}^{b_2} + (-2)\overbrace{(t+t^2)}^{b_3}$$

$$= 5 + 5t \quad + \quad 1 + t^2 - 2t - 2t^2$$

$$= 6 + 3t - t^2$$

Ex $V = \mathbb{P}_2$, with standard basis $\mathcal{B} = \{1, t, t^2\}$

Then $p = 6(1) + 3(t) - 1(t^2)$, so

the coordinate vector of p relative to the standard basis

is $[p]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}$.

II. How to find $\mathcal{B}$-coordinate vector of an element in $\mathbb{R}^n$

Ex Let $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{b}_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{b}_2} \right\}$. This is a basis for $\mathbb{R}^2$.

Find $[\vec{x}]_{\mathcal{B}}$, the coordinate vector of $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ relative to $\mathcal{B}$.

Sol: The $\mathcal{B}$-coordinates c_1, c_2 of $\vec{x}$ are solutions to

$$c_1 \vec{b}_1 + c_2 \vec{b}_2 = \vec{x}$$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

This matrix $P_{\mathcal{B}}$ whose columns are $\vec{b}_1, \vec{b}_2$ is called

the <u>change-of-coordinates matrix</u> from $\mathcal{B}$

to the standard basis $\{\vec{e}_1, \vec{e}_2\}$ in $\mathbb{R}^2$,

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & 1 & 5 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}$$

OR

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{4+5}{3} \\ \frac{-4+10}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

So $\vec{x} = 3\vec{b}_1 + 2\vec{b}_2$

thus $\boxed{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$

Algorithm for finding change-of-coordinates matrix from $\mathcal{B}$ to the standard basis for $\mathbb{R}^n$

In general: Fix a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ for $\mathbb{R}^n$.

Construct an $n \times n$ matrix $P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$,

called the <u>change-of-coordinates matrix</u> from basis $\mathcal{B}$ to the standard basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ in $\mathbb{R}^n$.

Then the equation

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

is equivalent to

$$\boxed{\vec{x} = P_{\mathcal{B}}\,[\vec{x}]_{\mathcal{B}}} \quad \text{aka} \quad \vec{x} = \underbrace{\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix}}_{P_{\mathcal{B}}}\underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}_{[\vec{x}]_{\mathcal{B}}}$$

To solve for $[\vec{x}]_{\mathcal{B}}$

possible way #1:

$$\underbrace{\left[\begin{array}{c|c} P_{\mathcal{B}} & \vec{x} \end{array}\right]}_{\text{Augmented matrix}} \xrightarrow{\text{Row reduce}} \left[\begin{array}{cccc|c} 1 & & & & c_1 \\ & 1 & & & c_2 \\ & & \ddots & & \vdots \\ & & & 1 & c_n \end{array}\right]$$

possible way #2:

Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^n$,
$P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Thm)

Multiplying both sides on the left by $P_{\mathcal{B}}^{-1}$:

$$P_{\mathcal{B}}^{-1}\vec{x} = P_{\mathcal{B}}^{-1} P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$$

$$\boxed{P_{\mathcal{B}}^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}} \quad \text{aka} \quad \begin{bmatrix} c_1 \\ \vdots \\ c_2 \end{bmatrix} = P_{\mathcal{B}}^{-1}\vec{x}$$

II. Coordinate mapping / isomorphism of vector spaces

Def The map $T: \mathbb{R}^n \to \mathbb{R}^n$

$$\vec{x} \longmapsto P_{\mathcal{B}}^{-1}\vec{x}$$

(equivalently, $\vec{x} \longmapsto [\vec{x}]_{\mathcal{B}}$)

is called the coordinate mapping (determined by $\mathcal{B}$)

Fact The coordinate mapping is a linear map $\mathbb{R}^n \to \mathbb{R}^n$ which is one-to-one and onto.

Proof Since the standard matrix of this map is $P_{\mathcal{B}}^{-1}$ which is an invertible $n \times n$ matrix, it follows from the Invertible matrix Theorem that this map is one-to-one and onto.

Ex (again) $V = \mathbb{P}_2$, with standard basis $\mathcal{B} = \{1, t, t^2\}$

The coordinate mapping $T: V \longrightarrow \mathbb{R}^3$

$p \longmapsto [p]_{\mathcal{B}}$

is a linear map which is one-to-one and onto.

Def Given vector spaces V and W,

a linear map $T: V \rightarrow W$

is called a (vector space) isomorphism

if T is one-to-one and onto.

Note: If there is an isomorphism from V onto W,

it means that, although the notation and

terminology for V and W may differ,

V and W are indistinguishable as vector spaces.

$\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$.

Thm 9 Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis for a vector space V. Then the coordinate mapping $x \mapsto [x]_{\mathcal{B}}$ is an isomorphism from V onto $\mathbb{R}^n$

This is a big deal! This means every vector space V is "the same" (as vector spaces) as $\mathbb{R}^n$ if V has a basis with n elements.

Ex 7 Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and let $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

- Since $\vec{v}_1$ and $\vec{v}_2$ are not scalar multiples of each other, we know $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

  So $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for H.

- By Thm 9, H is isomorphic to $\mathbb{R}^2$

(a) Now, determine whether $\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ is in H

Sol:

Check whether the equation $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{x}$ is consistent.

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_1 = 2 \\ c_2 = 3 \end{array}$$

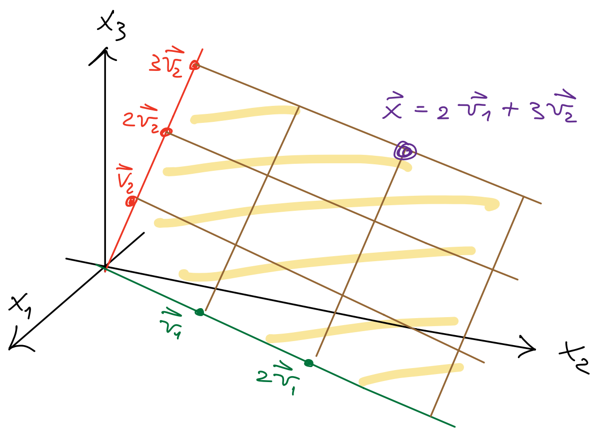
So $\vec{x}$ $\underline{\underline{\text{is}}}$ in H: $\vec{x} = 2\vec{v}_1 + 3\vec{v}_2$.

(b) Find the Coordinate vector of $\vec{x}$ relative to $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$

Sol: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Picture

Basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ imposes a coordinate system ("graph paper") on the plane H in $\mathbb{R}^3$



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